Equivariant localization, parity sheaves, and cyclic base change

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Global Langlands correspondence

Notation:

- F = global function field, e.g. $\mathbb{F}_\ell(t)$
- G = reductive group over F, e.g. SL_n
- $k = \overline{\mathbb{F}}_p$ (coefficients), $p \neq \mathrm{char}(F)$

Vincent Lafforgue constructed

$$\left\{ \begin{array}{c} \text{irreducible cuspidal} \\ \text{automorphic representations} \\ \text{of } G \text{ over } k \end{array} \right\} \to \left\{ \begin{array}{c} \text{Langlands parameters} \\ \mathrm{Gal}(F^s/F) \to {}^L G(k)/\sim \end{array} \right\}.$$

Does it have expected properties?

Langlands functoriality:

$$
\begin{tikzcd}
f_H \arrow[r, squiggly] \arrow[d, squiggly] & \rho_f \arrow[d] & {}^L H \arrow[d, "\phi"] \\
f_G \arrow[r, squiggly] & \phi \circ \rho_f & {}^L G
\end{tikzcd}
$$

$$\left\{\begin{matrix}\text{irreducible}\\ \text{automorphic representations}\\ \text{of } G \text{ over } k\end{matrix}\right\} \to \left\{\begin{matrix}\text{Langlands parameters}\\ \mathrm{Gal}(F^s/F) \to {}^L G(k)/\sim\end{matrix}\right\}.$$

ρ

Global base change functoriality: Suppose

- H reductive over F,
- E/F field extension, $G := \mathrm{Res}_{E/F}(H_E)$.

$$\begin{tikzcd} \phi : {}^L H \arrow[r] & {}^L G \\ \hat{H} \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "\Delta"] & \hat{G} \arrow[u, phantom, "\cup" rotate=90] = \hat{H}^{[E:F]} \end{tikzcd}$$

$p > 5$ ok?

+ conditional on Xue

Thm* $[E:F] = p$, cyclic. p odd, good for $\hat{G}$.

Then ρ_H automorphic $\Rightarrow$ $\phi \circ \rho_H$ autom.

Previous proofs of base change (for GL_n) are based on the trace formula.

(e.g. SL_n $n > 2$)

Novelty for general G: can have

f, f' generating *isomorphic* automorphic representations $\to$ different L-parameters.

π_f, $\pi_{f'}$

Indistinguishable by the trace formula!

Local Langlands correspondence

Notation:

- F_v = local function field of $\mathrm{char} \neq p$, e.g. $\mathbb{F}_\ell((t))$.
- H = reductive group over F_v.

Genestier-Lafforgue constructed:

$$\left\{ \begin{matrix} \text{irreducible representations} \\ \text{of } H(F_v) \text{ over } k \end{matrix} \right\} / \sim \longrightarrow \left\{ \begin{matrix} \text{semi-simple} \\ \text{Langlands parameters} \\ \mathrm{Weil}(F_v) \to {}^L H(k) / \sim \end{matrix} \right\}.$$

Does it have expected properties?

We will investigate local base change:

- E_v/F_v extension, $\mathrm{Gal}(E_v/F_v) \approx \mathbb{Z}/p = \langle \sigma \rangle$.
- $G = \mathrm{Res}_{E_v/F_v}(H_{E_v})$

ϕ

$$\sigma \circlearrowleft \left\{ \begin{matrix} \text{irreducible representations} \\ \text{of } H(F_v) \text{ over } k \end{matrix} \right\} \overset{\mathrm{BC}}{\dashrightarrow} \left\{ \begin{matrix} \text{irreducible representations} \\ \text{of } G(F_v) \text{ over } k \end{matrix} \right\}$$

$\pi \qquad \Pi$

$\sigma \curvearrowright \mathrm{Rep}(G) \ni \Pi \mapsto \Pi^\sigma : \quad G \xrightarrow{\sigma} G \xrightarrow{\Pi} \mathrm{End}(\Pi)$

σ-fixed if $\Pi \approx \Pi^\sigma$

Lemma $\Longrightarrow$ $\Pi \circlearrowleft G$, $\quad G \rtimes \sigma$

$\mathrm{BC}^{-1}(\Pi) = ?$

Tate cohomology

$G \quad G \curvearrowright \Pi \circlearrowleft (\mathbb{Z}/p\mathbb{Z}) = \langle \sigma \rangle$

$$0 = \sigma^p - 1 = (\sigma - 1)\underbrace{(1 + \sigma + \cdots + \sigma^{p-1})}_{N} \in \mathbb{Z}[\sigma]$$

$\curvearrowright G^\sigma = H$

$$T^0(\Pi) = \frac{\Pi^\sigma}{N \cdot \Pi}$$

$$\Pi \xrightarrow{N} \Pi \xrightarrow{1-\sigma} \Pi \xrightarrow{N} \Pi \xrightarrow{1-\sigma} \cdots$$

$$T^1(\Pi) = \frac{\ker(N)}{\mathrm{Im}\,(\sigma - 1)}$$

Treumann-Venkatesh Conjecture

Conjecture (Treumann-Venkatesh)

Let Π be an irreducible σ-fixed representation of $G(F_v)$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under LLC to

$$\Pi^{(p)} := \Pi \otimes_{k,\mathrm{Frob}_p} k.$$

Genestier-Lafforgue,

Ex $G = \overset{\sigma}{\curvearrowright} H^p \; (= \mathrm{Res}_{F_v^p/F_v} H)$

$H \curvearrowright \pi \rightsquigarrow G \curvearrowright \Pi = \pi \otimes \cdots \otimes \pi$

$v \in \pi$

$N \cdot (v \otimes w \otimes 2)$

$$T^0(\Pi) = \frac{\Pi^\sigma}{N \cdot \Pi} = \langle v \otimes v \otimes v,\ v \otimes w \otimes 2 + w \otimes 2 \otimes v + 2 \otimes v \otimes w \rangle$$

$\pi^{(p)}$

$$X = \frac{(1+\sigma + \dots + \sigma^{p-1})x}{p}$$

$G \qquad H = G^{\mathbb{Z}/p\mathbb{Z}}$

$\longrightarrow$ analogous results

$\left[Bm_G^{\sigma} = Bm_H \right]$

$\underline{\text{ex}}$ act ind $GL_n \qquad GL_{np}$

Treumann-Venkatesh Conjecture

Conjecture (Treumann-Venkatesh)

Let Π be an irreducible σ-fixed representation of $G(F_v)$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under LLC to

$$\Pi^{(p)} := \Pi \otimes_{k,\mathrm{Frob}_p} k.$$

Theorem (F.)

Assume p is odd and good for $\widehat{G}$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under the Genestier-Lafforgue correspondence to $\Pi^{(p)}$.

$$T^i(\Pi) \subset BC^{-1}(\Pi^{(p)})$$

- Previously proved by Ronchetti for depth zero supercuspidals of GL_n induced from cuspidal Deligne-Lusztig representations.

Plan

1. Statement of the results. ✓
2. Summary of Lafforgue's idea.
3. Equivariant localization.
4. Modular representation theory.

The excursion algebra

Let Γ be a group, $\widehat{G}$ a reductive group over k.

comm $\mathrm{Exc}(\Gamma, \hat{G}) \underset{\text{Drinfeld, Zhu}}{\sim}$ "functions on $\{\Gamma \to \hat{G} /\sim\}$"

Key property $\quad x_f \longleftrightarrow \rho_f$

$$\left\{ \begin{array}{c} \text{chars } \mathrm{Exc}(\Gamma, \hat{G}) \\ \to k \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{semisimple} \\ \Gamma \to \hat{G}/\sim \end{array} \right\}$$

Want construct $\mathrm{Exc}(\Gamma, \hat{G}) \curvearrowright \{\text{aut functions}\}$

$$\mathrm{Exc}(\Gamma, \hat{G}) \xrightarrow{x_f} k \qquad \qquad \{\text{aut functions}\} \ni f$$

Can present $\mathrm{Exc}(\Gamma, \widehat{G})$ explicitly by generators and relations.

Generators: $\underbrace{S_{n,f,(\gamma_i)_{i=1,\ldots,n}}}$

- $n \geq 0$
- $f \in \mathcal{O}(\hat{G} \backslash \hat{G}^n / \hat{G})$
- $\gamma_1, \ldots, \gamma_n \in \Gamma$

"Value" at $\rho : \Gamma \to \hat{G}/\sim$ is

$$f(\rho(\gamma_1\gamma_n, \ldots, \gamma_{n-1}\gamma_n, \gamma_n)) \in k$$

Relations are complicated:

4.2.2. *Relations.* Next we describe the relations. (Compare [Laf18a, §10].)

(i) $S_{\emptyset,f,*} = f(1_G)$.

(ii) The map $f \mapsto S_{I,f,(\gamma_i)_{i\in I}}$ is a k-algebra homomorphism in f, i.e.

$$S_{I,f+f',(\gamma_i)_{i\in I}} = S_{I,f,(\gamma_i)_{i\in I}} + S_{I,f',(\gamma_i)_{i\in I}},$$
$$S_{I,ff',(\gamma_i)_{i\in I}} = S_{I,f,(\gamma_i)_{i\in I}} \cdot S_{I,f',(\gamma_i)_{i\in I}},$$

and

$$S_{I,\lambda f,(\gamma_i)_{i\in I}} = \lambda S_{I,f,(\gamma_i)_{i\in I}} \text{ for all } \lambda \in k.$$

(iii) For all maps of finite sets $\zeta\colon I \to J$, all $f \in \mathcal{O}(\widehat{G}_k\backslash({}^LG_k^{\mathrm{alg}})^I/\widehat{G}_k)$, and all $(\gamma_j)_{j\in J} \in \Gamma^J$, we have

$$S_{J,f^\zeta,(\gamma_j)_{j\in J}} = S_{I,f,(\gamma_{\zeta(i)})_{i\in I}}$$

where $f^\zeta \in \mathcal{O}(\widehat{G}_k\backslash({}^LG_k^{\mathrm{alg}})^J/\widehat{G}_k)$ is defined by $f^\zeta((g_j)_{j\in J}) := f((g_{\zeta(i)})_{i\in I})$.

(iv) For all $f \in \mathcal{O}(\widehat{G}_k\backslash({}^LG_k^{\mathrm{alg}})^I/\widehat{G}_k)$ and $(\gamma_i)_{i\in I}, (\gamma_i')_{i\in I}, (\gamma_i'')_{i\in I} \in \Gamma^I$, we have

$$S_{I\sqcup I\sqcup I,\widetilde{f},(\gamma_i)_{i\in I}\times(\gamma_i')_{i\in I}\times(\gamma_i'')_{i\in I}} = S_{I,f,(\gamma_i(\gamma_i')^{-1}\gamma_i'')_{i\in I}},$$

where $\widetilde{f} \in \mathcal{O}(\widehat{G}_k\backslash({}^LG_k^{\mathrm{alg}})^{I\sqcup I\sqcup I}/\widehat{G}_k)$ is defined by

$$\widetilde{f}((g_i)_{i\in I} \times (g_i')_{i\in I} \times (g_i'')_{i\in I}) = f((g_i(g_i')^{-1}g_i'')_{i\in I}).$$

Actions of the excursion algebra

How to construct $\mathrm{Exc}(\Gamma, \widehat{G}) \curvearrowright V$

Tannakian construction: Given

family of functors

$$\begin{tikzcd}
\mathrm{Rep}(\hat{G}^I) \arrow[r, "H_I"] \arrow[d] & \mathrm{Rep}(\Gamma^I) \arrow[d] \\
\mathrm{Rep}(\hat{G}^J) \arrow[r, "H_J"] \arrow[d] & \mathrm{Rep}(\Gamma^J) \arrow[d] \\
\mathrm{Rep}(\hat{G}^K) \arrow[r, "H_K"] & \mathrm{Rep}(\Gamma^K)
\end{tikzcd}$$

fin. set $I \to J \to K$

~~$\oplus$ $\rho: \Gamma \to \hat{G}$~~

Output

action of $\mathrm{Exc}(\Gamma, \hat{G})$

on $V = H_{\{1\}}(\mathbb{1})$

$\shortparallel$

$H_{\emptyset}(\mathbb{1})$

Summary of Lafforgue's correspondence

<u>Where does this structure come from?</u>

$F \leftrightarrow$ X smooth projective curve.

$\Gamma := \pi_1(X)$

$$\mathrm{Rep}(\hat{G}^I) \xrightarrow{\mathrm{Sat}} \left\{ \text{sheaves on } \mathrm{Sht}_I \right\}$$

Γ^I

$$W \longmapsto R\pi_!\left(\mathrm{Sht}_I, \mathrm{Sat}(W)\right)$$

$$\pi_1(X^I) \longrightarrow \pi_1(X)^I$$

$$\mathrm{Sht}_I \xrightarrow{\pi_I} X^I$$

Summary of Lafforgue's correspondence

Where does this structure come from?

Geometric Satake equivalence:

perverse

$$\underbrace{\mathrm{Rep}_k(\widehat{G})} \cong P_{G(\mathcal{O}_v)}(\underbrace{G(F_v)/G(\mathcal{O}_v)}_{\text{“affine Grassmannian” } \mathrm{Gr}_G}).$$

Viewed as sheaves on moduli spaces of shtukas:

$$\mathrm{Sht}_I = \left\{ \begin{array}{l} \mathcal{E} = G\text{-bundle} / X \\ \{x_i\}_{i \in I} \subset X \\ \mathcal{E}|_{X - \{x_i\}} \xrightarrow{\sim} \mathrm{Fr}^* \mathcal{E}|_{X - \{x_i\}} \end{array} \right\}$$

$$\mathrm{Sht}_I \xrightarrow{\pi_I} X^I$$

"aut. functions"

$$\mathrm{Exc}(\Pi, \vec{a}) \curvearrowright H_c^*(\underbrace{\mathrm{Sht}_{\phi}}_{u}, \underline{k}) = \mathrm{Fun}^c(\mathrm{Bun}_G(\mathbb{F}_\ell))$$

$$\mathrm{Bun}_G(\mathbb{F}_\ell) \overset{\mathrm{Weil}}{\sim} G(F) \backslash G(\mathbb{A}_F) / G(\mathcal{O})$$

Summary

- Source of Galois representations: cohomology of moduli spaces of shtukas. (Topology)
- Excursion operators: endomorphisms of automorphic forms coming from "combinatorial" pattern of maps between cohomology groups.
- Langlands parametrization comes from having sheaves indexed by $\mathrm{Rep}(\widehat{G})$. $\widehat{G}$

(Rep. Theory)

Suppose we want Langlands functoriality between H and G:

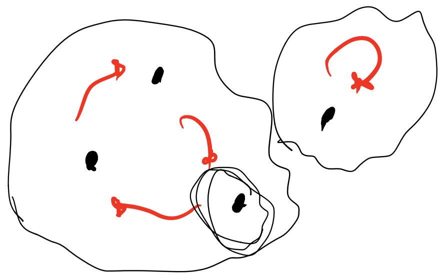
- (Topology) Need mechanism to relate cohomology of shtukas for G and for H $\leftarrow$ equivariant localization.
- (Representation Theory) Need mechanism to relate sheaves indexed by $\mathrm{Rep}(\widehat{G})$ and by $\mathrm{Rep}(\widehat{H})$ $\leftarrow$ sheaf-theoretic Smith theory.

Equivariant localization

Equivariant localization $\implies$ relationship between the cohomology of a space and its fixed point subspace under a group action.

Ex $\quad G \circlearrowright S \quad$ set

$\mathbb{Z}/p \cong \langle \sigma \rangle$



$$\boxed{T^0(\mathrm{Fun}^c(S, k))} = \frac{\mathrm{Fun}^c(S,k)^\sigma}{N \cdot \mathrm{Fun}^c(S,k)} \simeq \boxed{\mathrm{Fun}^c(S^\sigma)} \cancel{\otimes N},$$

σ

Base change situation: $G = \mathrm{Res}_{E/F}(H_E)$.

$$S = \mathrm{Bun}_G(\mathbb{F}_\ell) \qquad S^\sigma \simeq \mathrm{Bun}_H(\mathbb{F}_\ell)$$

$$\begin{tikzcd}
\mathrm{Hk}(G)^\sigma \arrow[rr, dashed] \arrow[d, phantom, "\circlearrowright"] \arrow[rr, phantom, bend right, "{[\mathrm{Tr}]}"'] & & \mathrm{Hk}(H) \arrow[d, phantom, "\circlearrowright"] \\
T^0 \mathrm{Fun}^c(\mathrm{Bun}_G(\mathbb{F}_\ell)) \arrow[rr, phantom, "\simeq"] & & \mathrm{Fun}^c(\mathrm{Bun}_H(\mathbb{F}_\ell)) \\
\mathrm{Exc}(\Gamma, {}^L G)^\sigma \arrow[u, phantom, "\circlearrowright"] \arrow[rr, dashed] & & \mathrm{Exc}(\Gamma, {}^L H) \arrow[u, phantom, "\circlearrowright"]
\end{tikzcd}$$

$\psi \ni f_G \;\text{- - -}\; f_H \in \psi$

Hope

$\mathrm{Exc}(\Gamma, {}^L G)^\sigma \supseteq$

Prove: $\mathrm{Exc}(\Gamma, {}^L G)'$ big enough.

For global base change, we need to transfer eigensystems for the excursion operators.

- Need to study possible extensions of a character of $\mathrm{Exc}(\Gamma, \widehat{G})'$ to all of $\mathrm{Exc}(\Gamma, \widehat{G})$.

Lemma $\exists!$ extension.

For local base change, we need to examine the construction of the Genestier-Lafforgue correspondence.

- Study $S_{n,f,(\gamma_i)}$ for $\{\gamma_i\} \subset \mathrm{Weil}(\overline{F}_v/F_v) \subset \mathrm{Gal}(\overline{F}/F)$.

Details omitted here.

Equivariant localization and excursion operators

$W \in \mathrm{Rep}(\hat{G})$

Excursion operators:

$$\begin{tikzcd}
G & T^0 H^0_c(\mathrm{Sht}_G; \mathrm{Sat}(\mathbb{1})) \arrow[r] \arrow[d, equal] & T^0(\mathrm{Sht}_G; \mathrm{Sat}(W)) \arrow[r] \arrow[d, dashed, "?"] & \ldots \arrow[d, dashed, "?"] \\
H & T^0 H^0_c(\mathrm{Sht}_H; \mathrm{Sat}(\mathbb{1})) \arrow[r] & T^0(\mathrm{Sht}_H, \mathrm{Sat}(\mathrm{Res}(W))) \arrow[r] & \ldots
\end{tikzcd}$$

H_c^0

Want to identify all steps of the excursion. Lemma: $\mathrm{Sht}_G^\sigma \simeq \mathrm{Sht}_H$

- **Topological** aspect: relate (Tate) cohomology of a space with (Tate) cohomology of its fixed points.
- **Representation-theoretic** aspect: geometric interpretation of restriction functor $\mathrm{Rep}_k({}^L G) \xrightarrow{\mathrm{Res}} \mathrm{Rep}_k({}^L H)$.

Tate cohomology

Suppose $\langle\sigma\rangle \approx \mathbb{Z}/p \curvearrowright X$, $\mathcal{F} \in D^b_\sigma(X;k)$.

$$\sigma \circlearrowright \underline{C^\bullet(X, \mathcal{F})}$$

$T^i(X;\mathcal{F}) :=$

$$H^i(\mathrm{Tot}(\ldots \xrightarrow{N} C^*(X;\mathcal{F}) \xrightarrow{1-\sigma} C^*(X;\mathcal{F}) \xrightarrow{N} C^*(X;\mathcal{F}) \xrightarrow{1-\sigma} \ldots))$$

$$M^{\circlearrowright\sigma} \quad \xrightarrow{N} M \xrightarrow{1-\sigma} M \xrightarrow{N}$$

Equivariant localization (Smith, Quillen, Treumann)

$$T^i(X;\mathcal{F}) \cong T^i(X^\sigma;\mathcal{F}|_{X^\sigma}).$$

Can apply this to shtukas because (Lemma): $\mathrm{Sht}_G^\sigma = \mathrm{Sht}_H$.

$$X = X^\sigma \cup (X - X^\sigma)$$

σ acts freely

$$C^*(X - X^\sigma)$$

perfect over $k[\sigma]$

$$\downarrow T^*(-)$$

$$0$$

Tate cohomology kills perfect complexes.

Summary

- We have explained the topological input into functoriality.
- Now zoom in on the representation-theoretic input:

$$
\begin{tikzcd}
\text{Hecke category for } G \arrow[r, dashed] \arrow[d, equal] & \text{Hecke category for } H \arrow[d, equal] \\
\mathrm{Rep}_k({}^L G) \arrow[r] & \mathrm{Rep}_k({}^L H)
\end{tikzcd}
$$

Smith theory

Notation:

- H = reductive group over F_v.
- E_v/F_v extension, $\mathrm{Gal}(E_v/F_v) \approx \mathbb{Z}/p = \langle \sigma \rangle$.
- $G = \mathrm{Res}_{E_v/F_v}(H_{E_v})$
- (Coefficients) $k = \overline{\mathbb{F}}_p$.

Treumann-Venkatesh construct Brauer homomorphism

$$\begin{array}{ccc} k\text{-Hecke } \underline{\text{algebra}} \text{ for } G & \xrightarrow{\text{br}} & k\text{-Hecke } \underline{\text{algebra}} \text{ for } H \\ \text{Sat isom} \quad \Vert & & \Vert \\ K_0 \operatorname{Rep}_k({}^L G) & \xrightarrow{\text{Res}} & K_0 \operatorname{Rep}_k({}^L H) \end{array}$$

With Gus Lonergan we construct a *categorification*

$$\begin{array}{ccc} k\text{-Hecke } \underline{\text{category}} \text{ for } G & \xrightarrow{\text{BC}} & k\text{-Hecke } \underline{\text{category}} \text{ for } H \\ \text{Geom. Sat } \Big\| & & \Big\| \\ \mathrm{Rep}_k({}^L G) & \xrightarrow{\quad\text{Res}\quad} & \mathrm{Rep}_k({}^L H) \end{array}$$

using recent tools in geometric representation theory:

- parity sheaves (Juteau-Mautner-Williamson),
- Smith-Treumann theory (Treumann, Leslie-Lonergan, Riche-Williamson).

Brauer homomorphism

Assume: $[\frac{G(F_v)}{G(\mathcal{O}_v)}]^\sigma = [\frac{H(F_v)}{H(\mathcal{O}_v)}] = [H]$

$[G]^\sigma =$

k-Hecke algebra for $G \xrightarrow{\text{br}} k$-Hecke algebra for H

$$\mathrm{Fun}^c_{G(F_v)}(\frac{G(F_v)}{G(\mathcal{O}_v)} \times \frac{G(F_v)}{G(\mathcal{O}_v)}, k) \longrightarrow \mathrm{Fun}^c_{H(F_v)}(\frac{H(F_v)}{H(\mathcal{O}_v)} \times \frac{H(F_v)}{H(\mathcal{O}_v)}, k)$$

$f \mapsto f * {}^\sigma f * \dots * {}^{\sigma^{p-1}} f \in \mathcal{H}_k(G)^\sigma$

$\phi \mapsto \phi|_{[H]^2}$

$$\sum_{y \in [G]} \phi|_{[H]^2}(x, y) \; \phi'|_{[H]^2}(y, z)$$

$y \in [H]$

$y \in [H]$

$$\sum_{y \in [a]-[H]} \phi(x, y) \, \phi'(y, z) = 0$$

$$\phi(x, y) = \phi(x, \sigma y) \qquad \phi'(y, z) = \phi'(\sigma y, z)$$

σ acts freely

[$*$ Fr^{-1} linearize]

Base change functor

$$k\text{-Hecke category for } G \xrightarrow{\text{BC}} k\text{-Hecke category for } H$$

$$P_{G(\mathcal{O}_v)}(\mathrm{Gr}_G) \dashrightarrow P_{H(\mathcal{O}_v)}(\mathrm{Gr}_H)$$

$\mathcal{F} \mapsto$

$$P_{G(\mathcal{O}_v)}(\mathrm{Gr}_G)^{\sigma} \xrightarrow{*\text{-res}} D_{H(\mathcal{O}_v)}(\mathrm{Gr}_H)^{\sigma}$$

$\cap$

$!$

$$\mathcal{F} * {}^{\sigma}\mathcal{F} * \dots * {}^{\sigma^{p-1}}\mathcal{F}$$

$D(\ )/\mathrm{Perf}$

$$D_{\sigma}(X^{\sigma}, k) \simeq D(X^{\sigma}, k[\sigma]) \supset \mathrm{Perf}(k[\sigma])$$

Defn Tate cat of X^σ = $\boxed{D_\sigma(X^\sigma, k) \big/ \mathrm{Perf}(k[\sigma])}$

$$\begin{tikzcd}
D_\sigma(X) \arrow[r, shift left, "*-\mathrm{res}"] \arrow[r, shift right, "!\,\mathrm{res}"'] \arrow[dr, bend right] & D_\sigma(X^\sigma) \arrow[d] \\
& D_\sigma(X^\sigma)/\mathrm{Perf}
\end{tikzcd}$$

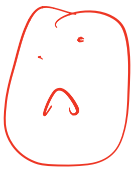
Ex $X^\sigma = pt.$

$$k \to k[\cancel{\sigma}] \to k[\cancel{\sigma}] \to k$$

$$\Rightarrow k \simeq k[2], \qquad \mathcal{F} \simeq \mathcal{F}[2]$$

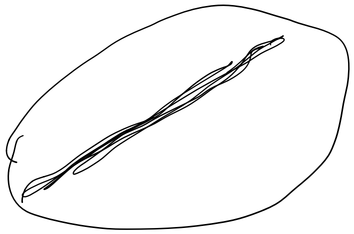
Perverse sheaves

- conditions on $*$ and $!$ stalks.
- cut out by $\leq, \geq$ on coh. degrees



Parity sheaves

- "
- congruences mod 2



Parity sheaves

Given a suitable stratification:

- (Juteau-Mautner-Williamson) Parity sheaves are $\mathcal{K} \in D^b$ whose $*$-stalks and !-stalks have cohomology concentrated either even degrees or odd degrees.
- (Leslie-Lonergan) Tate-Parity sheaves are $\mathcal{K} \in D^b/\mathrm{Perf}$ whose $*$-stalks and !-stalks have Tate cohomology concentrated in either even degrees or odd degrees.

Feature: (Tate-)Parity sheaves enjoy strong rigidity properties.

$$
\begin{tikzcd}
\mathrm{Parity}^0_{G(\mathcal{O})}(\mathrm{Gr}_G) \arrow[r] \arrow[ddr, bend right, "\mathrm{BC}"'] & \mathrm{Parity}^0_{G(\mathcal{O})\rtimes\sigma}(\mathrm{Gr}_G) \arrow[r, "\mathrm{Res}"] \arrow[d, "\text{Thm}"', "\text{restrict}"] & D_{H(\mathcal{O})\rtimes\sigma}(\mathrm{Gr}_H) \arrow[d] \\
 & \text{Tate-Parity}^0 \arrow[d, "\text{Thm}"', "\text{lift}"] \arrow[r, phantom, "\subset"] & D_{H(\mathcal{O})\rtimes\sigma}(\mathrm{Gr}_H)/\mathrm{Perf} \\
 & \mathrm{Parity}^0(\mathrm{Gr}_H) \arrow[r, phantom, "\subset"] & P_{H(\mathcal{O})}(\mathrm{Gr}_H)
\end{tikzcd}
$$

$$\mathcal{F} \longmapsto \mathcal{F} * {}^{\sigma}\mathcal{F} * \ldots * {}^{\sigma^{p-1}}\mathcal{F}$$

This is interesting even for $H = \mathrm{GL}(1)$, for which Gr_H is (étale homotopic to) a discrete union of points.

p good

Mautner-Rich

$$\begin{tikzcd} \mathrm{Parity}^{\circ}_{G(\mathcal{O})}(Gr_G) \arrow[r, phantom, "\subset"] \arrow[d, leftrightarrow] & P_{G(\mathcal{O})}(Gr_G) \arrow[d, leftrightarrow] \\ \underline{\mathrm{Tilt}(\hat{G})} \arrow[r, phantom, "\subset"] & \underline{\mathrm{Rep}(\hat{G})} \end{tikzcd}$$

<u>Fact</u> $\exists$ enough Tilting modules

Rmks k char 0 Parity ⓪ $\Leftrightarrow$ Perverse