Equivariant localization, parity sheaves, and cyclic base change

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Global Langlands correspondence

Notation:

- \( F \) = global function field, e.g. \( \mathbb{F}_\ell(t) \)
- \( G \) = reductive group over \( F \), e.g. \( \text{SL}_n \)
- \( k = \overline{\mathbb{F}}_p \) (coefficients), \( p \neq \text{char}(F) \)

Vincent Lafforgue constructed

\[
\left\{ \begin{array}{c}
\text{irreducible cuspidal} \\
\text{automorphic representations}
\end{array} \right\} \rightarrow \left\{ \begin{array}{c}
\text{Langlands parameters} \\
\text{Gal}(F^s/F) \rightarrow {}^L G(k)/\sim
\end{array} \right\}.
\]

Does it have expected properties?
Langlands functoriality:
Global base change functoriality: Suppose

- $H$ reductive over $F$,
- $E/F$ field extension, $G := \text{Res}_{E/F}(H_E)$.

\[
\phi : \hat{H} \rightarrow \hat{G}
\]

\[
\hat{H} \rightarrow \Delta \rightarrow \hat{G} \cong \hat{\text{Gal}(E:F)}
\]

Then $[E:F] = p$, cyclic, $p$ odd, good for $G$.

Then $\rho_H$ automorphic $\Rightarrow$ $\phi \circ \rho_H$ autom.
Previous proofs of base change (for GL$_n$) are based on the trace formula. (e.g. SL$_n$, $n \geq 2$)

Novelty for general $G$: can have

- $f, f'$ generating *isomorphic* automorphic representations
- $\Pi_{\xi}, \Pi_{\xi'}$ → different $L$-parameters.

Indistinguishable by the trace formula!
Local Langlands correspondence

Notation:
- $F_v = \text{local function field of } \text{char} \neq p$, e.g. $\mathbb{F}_\ell((t))$.
- $H = \text{reductive group over } F_v$.

Genestier-Lafforgue constructed:

\[
\left\{ \text{irreducible representations of } H(F_v) \text{ over } k \right\} / \sim \rightarrow \left\{ \overset{\text{semi-simple}}{\text{Langlands parameters}} \right. \\
\left. \text{Weil}(F_v) \rightarrow \overset{L}{H}(k)/ \sim \right\}.
\]

Does it have expected properties?
We will investigate local base change:

- $E_v/F_v$ extension, $\text{Gal}(E_v/F_v) \approx \mathbb{Z}/p = \langle \sigma \rangle$.
- $G = \text{Res}_{E_v/F_v}(H_{E_v})$

\[ \sigma \left\{ \begin{array}{l}
\text{irreducible representations of } H(F_v) \text{ over } k \\
\text{irreducible representations of } G(F_v) \text{ over } k
\end{array} \right\} \stackrel{BC}{\longrightarrow} \prod
\]

\[ \sigma \text{-fixed if } \pi \cong \pi^\sigma \]

\[ \text{Rep}(\mathfrak{a}) \ni \pi \mapsto \pi^\sigma : \mathfrak{a} \to \mathfrak{a} \to \text{End}(\pi) \]

\[ \text{BC}^{-1}(\pi) = ? \]
Tate cohomology

\[ G \rightarrow \pi D (\mathbb{I} / \mathbb{Z}) = \langle \sigma \rangle \]

\[ 0 = \sigma^{p-1} = (\sigma - 1)(1 + \sigma + \ldots + \sigma^{p-1}) \in \mathbb{Z} [\sigma] \]

\[ T^\sigma (\pi) = \frac{\pi^\sigma}{N \cdot \pi} \]

\[ T^\sigma (\pi) = \frac{\ker (N)}{\text{im} (\sigma - 1)} \]

\[ N \rightarrow \pi \rightarrow \pi \rightarrow N \]
Let $\Pi$ be an irreducible $\sigma$-fixed representation of $G(F_v)$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under LLC to

$$\Pi^{(p)} := \Pi \otimes_{k, \text{Frob}_p} k.$$
\[ X = \frac{(1 + \sigma \cdots \rho^{-1})}{\rho} \]

\[ G \uparrow H = G^{2/\rho_2} \]

\[ \text{analogous results} \]

\[ B_m^c = B_m^H \]

ex act and can allow
Conjecture (Treumann-Venkatesh)

Let $\Pi$ be an irreducible $\sigma$-fixed representation of $G(F_v)$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under LLC to

$$
\Pi^{(p)} := \Pi \otimes_{k, \text{Frob}_p} k.
$$

Theorem (F.)

Assume $p$ is odd and good for $\hat{G}$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under the Genestier-Lafforgue correspondence to $\Pi^{(p)}$.

Previously proved by Ronchetti for depth zero supercuspidals of $GL_n$ induced from cuspidal Deligne-Lusztig representations.
1. Statement of the results.
2. Summary of Lafforgue’s idea.
3. Equivariant localization.
4. Modular representation theory.
The excursion algebra

Let $\Gamma$ be a group, $\hat{G}$ a reductive group over $k$.

$\text{comm \ } \text{Exc}(\Gamma, \hat{G}) \sim \{ \text{functions on } \Gamma \} \leftarrow \{ \text{semi-simple } \Gamma \rightarrow \hat{G} / \sim \}$

Key property

$X_f \leftrightarrow \text{Pf}$

$
\{ \text{chars Exc}(\Gamma, \hat{G}) \} \leftrightarrow \{ \text{semi-simple } \Gamma \rightarrow \hat{G} / \sim \} \leftarrow \{ \text{aut functions} \}$

Want construct $\text{Exc}(\Gamma, \hat{G}) \leftrightarrow \{ \text{aut functions} \}$
Can present $\text{Exc}(\Gamma, \hat{G})$ explicitly by generators and relations.

Generators: $S_n, f, (\gamma_i)_{i=1,\ldots,n}$

- $n \geq 0$
- $f \in \mathbb{C}[a_1, \ldots, a_n]/\langle f \rangle$
- $\sigma_i, \tau_i \in \Gamma$

"Value" at $p \in \Gamma \mapsto \hat{\gamma}/\gamma$

$f \left( p \left( \sigma_1 \sigma_2 \ldots \tau_{n-1} \tau_n, \gamma \right) \right) \in \mathbb{C}$
Relations are complicated:

4.2.2. Relations. Next we describe the relations. (Compare [Laf18a §10].)

(i) \( S_{\emptyset, f, \ast} = f(1_G) \).

(ii) The map \( f \mapsto S_{I, f, (\gamma_i)_{i \in I}} \) is a \( k \)-algebra homomorphism in \( f \), i.e.

\[
S_{I, f+f', (\gamma_i)_{i \in I}} = S_{I, f, (\gamma_i)_{i \in I}} + S_{I, f', (\gamma_i)_{i \in I}},
\]

\[
S_{I, ff', (\gamma_i)_{i \in I}} = S_{I, f, (\gamma_i)_{i \in I}} \cdot S_{I, f', (\gamma_i)_{i \in I}},
\]

and

\[
S_{I, \lambda f, (\gamma_i)_{i \in I}} = \lambda S_{I, f, (\gamma_i)_{i \in I}} \quad \text{for all} \quad \lambda \in k.
\]

(iii) For all maps of finite sets \( \zeta: I \to J \), all \( f \in \mathcal{O}(\hat{G}_k \setminus (L_{G_k}^{\text{alg}})^{I} / \hat{G}_k) \), and all \( (\gamma_j)_{j \in J} \in \Gamma^J \), we have

\[
S_{J, f^\zeta, (\gamma_j)_{j \in J}} = S_{I, f, (\gamma_{\zeta(i)})_{i \in I}}
\]

where \( f^\zeta \in \mathcal{O}(\hat{G}_k \setminus (L_{G_k}^{\text{alg}})^{J} / \hat{G}_k) \) is defined by \( f^\zeta((g_j)_{j \in J}) := f((g_{\zeta(i)})_{i \in I}) \).

(iv) For all \( f \in \mathcal{O}(\hat{G}_k \setminus (L_{G_k}^{\text{alg}})^{I} / \hat{G}_k) \) and \( (\gamma_i)_{i \in I}, (\gamma'_i)_{i \in I}, (\gamma''_i)_{i \in I} \in \Gamma^I \), we have

\[
S_{I \sqcup I \sqcup I, \tilde{f}, (\gamma_i)_{i \in I} \times (\gamma'_i)_{i \in I} \times (\gamma''_i)_{i \in I}} = S_{I, f, (\gamma_i^{-1}_i)_{i \in I}}
\]

where \( \tilde{f} \in \mathcal{O}(\hat{G}_k \setminus (L_{G_k}^{\text{alg}})^{I \sqcup I \sqcup I} / \hat{G}_k) \) is defined by

\[
\tilde{f}((g_i)_{i \in I} \times (g'_i)_{i \in I} \times (g''_i)_{i \in I}) = f((g_i(g'_i)^{-1} g''_i)_{i \in I}).
\]
Actions of the excursion algebra

How to construct $\operatorname{Exc}(\Gamma, \hat{G}) \sim V$

Tannakian construction: Given

Family of functors

\[
\begin{align*}
\operatorname{Rep}(\hat{G})^I & \to \operatorname{Rep}(\hat{G})^I \\
\downarrow & \quad \downarrow \\
\operatorname{Rep}(\hat{G})^J & \to \operatorname{Rep}(\hat{G})^J \\
\downarrow & \quad \downarrow \\
\operatorname{Rep}(\hat{G})^K & \to \operatorname{Rep}(\hat{G})^K \\
\end{align*}
\]

Output action of $\operatorname{Exc}(\Gamma, \hat{G})$

on $V = H_{\text{eff}}(\Gamma)$

$H_{\text{eff}}(\Gamma)$
Summary of Lafforgue’s correspondence

Where does this structure come from?

\[ F \leftrightarrow X \text{ smooth projective curve.} \]

\[
\begin{array}{c}
\text{Rep}(\hat{\mathcal{A}}^I) \\
\text{Set} \\
\text{Sh}_{\mathcal{I}}(\text{Sh}_{\mathcal{I}}, \text{Set}(\mathcal{W})) \\
\text{C}_0 \\
\pi_c(X^I) \\
\pi_c(X)^I \\
\end{array}
\]

\[ \pi_c(X^I) \rightarrow \pi_c(X)^I \]
Summary of Lafforgue’s correspondence

Where does this structure come from?

Geometric Satake equivalence:

\[
\text{Rep}_k(\hat{G}) \cong P_G(\mathcal{O}_v) \left( \frac{G(F_v)/G(O_v)}{\{z\}} \right),
\]

"affine Grassmannian" $\text{Gr}_G$

Viewed as sheaves on moduli spaces of shtukas:
\[ \text{Exc}(\mathbb{N}, \mathbb{Z}) \cong \mathcal{H}_c^\infty (\text{Surf}_\mathcal{O}, k) = \text{Fun}_c (\text{B}m_c(\text{Fe})) \]

\[ \cong \text{Um}_c (\text{B}m_c(\text{Fe})) \sim \mathcal{C}(\mathbb{F}) \forall (\mathbb{A}_k) / (\mathbb{O}) \]
Summary

- **Source of Galois representations**: cohomology of moduli spaces of shtukas.
- **Excursion operators**: endomorphisms of automorphic forms coming from “combinatorial” pattern of maps between cohomology groups.
- **Langlands parametrization** comes from having sheaves indexed by $\text{Rep}(\hat{G})$.
Suppose we want Langlands functoriality between $H$ and $G$:

- **(Topology)** Need mechanism to relate cohomology of shtukas for $G$ and for $H \leftarrow$ **equivariant localization**.

- **(Representation Theory)** Need mechanism to relate sheaves indexed by $\text{Rep}(\hat{G})$ and by $\text{Rep}(\hat{H}) \leftarrow$ **sheaf-theoretic Smith theory**.
Equivariant localization

Equivariant localization $\rightarrow$ relationship between the cohomology of a space and its fixed point subspace under a group action.

\[ \mathbb{E}_X \subseteq S \Rightarrow \exists \sigma \in \langle \sigma \rangle \]

\[ \mathcal{I}^0 \left( \text{Fun}_c(S, k) \right) = \frac{\text{Fun}_c(S, k)^\sigma}{N \cdot \text{Fun}_c(S, k)} \]
Base change situation: $G = \text{Res}_{E/F}(H_E)$.

\[ S = \text{Bm}_a(F_E) \quad \quad S^\sigma = \text{Bm}_{H}(F_E) \]

\[ \text{Hc}(a)^g \quad \quad \Rightarrow \quad \text{Hc}(H) \]

\[ \text{proof:} \quad \text{Exc}(\mathbb{P},a)^e \quad \quad \quad \Rightarrow \quad \text{Exc}(\mathbb{P},H) \]

Prove: $\text{Exc}(\mathbb{P},a)^e$ big enough.
For global base change, we need to transfer eigensystems for the excursion operators.

- Need to study possible extensions of a character of \( \text{Exc}(\Gamma, \hat{G})' \) to all of \( \text{Exc}(\Gamma, \hat{G}) \).

**Lemma**: \( \exists ! \) extension.
For **local base change**, we need to examine the construction of the Genestier-Lafforgue correspondence.

- Study $S_{n,f,(\gamma_i)}$ for $\{\gamma_i\} \subset \text{Weil}(\overline{F}_v/F_v) \subset \text{Gal}(\overline{F}/F)$.

Details omitted here.
Equivariant localization and excursion operators

Excursion operators:

\[ T^0 H^0_c(Sht_G; \text{Sat}(I)) \to T^0(Sht_G; \text{Sat}(W)) \to \ldots \]

\[ T^0 H^0_c(Sht_H; \text{Sat}(I)) \to T^0(Sht_H; \text{Sat}(\text{Res}(W))) \to \ldots \]

Want to identify all steps of the excursion.

- **Topological** aspect: relate (Tate) cohomology of a space with (Tate) cohomology of its fixed points.

- **Representation-theoretic** aspect: geometric interpretation of restriction functor \( \text{Rep}_k(LG) \xrightarrow{\text{Res}} \text{Rep}_k(LH) \).
Suppose $\langle \sigma \rangle \cong \mathbb{Z}/p \cong X$, $\mathcal{F} \in D^b_{\sigma}(X; k)$.

$$T^i(X; \mathcal{F}) := H^i(\text{Tot}(\ldots \xrightarrow{N} C^*(X; \mathcal{F}) \xrightarrow{1-\sigma} C^*(X; \mathcal{F}) \xrightarrow{N} C^*(X; \mathcal{F}) \xrightarrow{1-\sigma} \ldots))$$
Equivariant localization (Smith, Quillen, Treumann)

\[ T^i(X; F) \cong T^i(X^\sigma; F|_{X^\sigma}). \]

Can apply this to shtukas because (Lemma): \( \text{Sht}^G = \text{Sht}_H. \)

\[ X = X^\sigma \cup (X - X^\sigma) \]

\[ \sim \quad \sim \]

\[ C^*(X - X^\sigma) \text{ ? } K^0T \]

\[ \{ \tau^*(\cdot) \}

\[ 0 \]

Tate cohomology kills perfect complexes.
We have explained the topological input into functoriality.

Now zoom in on the representation-theoretic input:

\[
\begin{array}{c}
\text{Hecke category for } G \quad \longrightarrow \quad \text{Hecke category for } H \\
\| \quad \| \\
\text{Rep}_k(LG) \quad \longrightarrow \quad \text{Rep}_k(LH)
\end{array}
\]
Smith theory

Notation:
- $H$ = reductive group over $F_v$.
- $E_v/F_v$ extension, $\text{Gal}(E_v/F_v) \cong \mathbb{Z}/p = \langle \sigma \rangle$.
- $G = \text{Res}_{E_v/F_v}(H_{E_v})$
- (Coefficients) $k = \overline{F}_p$.

Treumann-Venkatesh construct Brauer homomorphism

$k$-Hecke algebra for $G$ $\rightarrow$ $k$-Hecke algebra for $H$

$\text{Rep}_k(LG)$ $\rightarrow$ $\text{Rep}_k(LH)$
With Gus Lonergan we construct a categorification

\[ k\text{-Hecke category for } G \xrightarrow{\text{BC}} k\text{-Hecke category for } H \]

\[ \text{Geom. Sat} \]

\[ \text{Rep}_k(\mathcal{L}G) \xrightarrow{\text{Res}} \text{Rep}_k(\mathcal{L}H) \]

using recent tools in geometric representation theory:

- parity sheaves (Juteau-Mautner-Williamson),
- Smith-Treumann theory (Treumann, Leslie-Lonergan, Riche-Williamson).
Brauer homomorphism

Assume: \( \left[ \frac{G(F_v)}{G(O_v)} \right]^{\sigma} = \left[ \frac{H(F_v)}{H(O_v)} \right] = \mathcal{H} \)

\( \mathcal{H}^{\sigma} \quad k\text{-Hecke algebra for } G \xrightarrow{\text{br}} k\text{-Hecke algebra for } H \)

\( \text{Fun}^c_{G(F_v)} \left( \frac{G(F_v)}{G(O_v)} \times \frac{G(F_v)}{G(O_v)}, k \right) \xrightarrow{\phi} \text{Fun}^c_{H(F_v)} \left( \frac{H(F_v)}{H(O_v)} \times \frac{H(F_v)}{H(O_v)}, k \right) \)

\[ \sum_{y_0 \in \mathcal{L}_v^{\mathcal{H}} \setminus \mathcal{H}^2} \phi |(x, y) \quad \phi^{-1}|(y, 2) \]
$\sum_{y \in [\omega - [47]} \phi(x, y) \quad \phi(y, 2) = 0$

$\phi(x, 0y) \quad \phi'(s_0y, 2)$

acts freely

[* Fr$^{-1}$ linearize ]
$k$-Hecke category for $G$ \( \xrightarrow{\text{BC}} \) $k$-Hecke category for $H$

\[ P_G(\mathcal{O}_v)(\text{Gr}_G) \xrightarrow{\text{res}} P_H(\mathcal{O}_v)(\text{Gr}_H) \]

\[ D^\sigma_{\text{res}}(X^\sigma_{1,k}) \simeq D(\tau, k[0,1]) \subset \text{Perf}(k[0,1]) \]
Define \( \text{Tate cat of } X^\sigma = D_\sigma (x^\sigma, x) / \text{Perf}(k^\text{alg}) \)

\[
\begin{align*}
D_\sigma (x) & \xrightarrow{\text{res}} D_\sigma (x^\sigma) \\
\text{res} & \quad \downarrow \\
\downarrow & \quad \text{res} \\
D_\sigma (x^\sigma) & \xrightarrow{\text{perf}} \\
\end{align*}
\]

Example: 

\[
\begin{align*}
X^\sigma &= \text{pf.} \\
\therefore k &\rightarrow k[x^\sigma] \hookrightarrow k[x^\sigma] \rightarrow k \\
&\Rightarrow k = k[x^\sigma], \quad x^\sigma = x [x^\sigma]
\end{align*}
\]
Perverse sheaves

- Conditions on \* and \! stalks.

- Cut out by \leq, \geq on cohom. degrees

Parity sheaves

- Congruences mod 2
Parity sheaves

Given a suitable stratification:

- (Juteau-Mautner-Williamson) **Parity sheaves** are $\mathcal{K} \in D^b$ whose $\ast$-stalks and $!$-stalks have cohomology concentrated either in even degrees or odd degrees.

- (Leslie-Lonergan) **Tate-Parity sheaves** are $\mathcal{K} \in D^b/\text{Perf}$ whose $\ast$-stalks and $!$-stalks have Tate cohomology concentrated in either even degrees or odd degrees.

**Feature:** (Tate-)Parity sheaves enjoy strong rigidity properties.
Parity$^0_{G(O)(\text{Gr}_G)}$ \xrightarrow{\text{Res}} \text{Parity}^0_{G(\mathcal{O} \times \sigma)(\text{Gr}_G)} \xrightarrow{\sigma} D_{H(\mathcal{O} \times \sigma)(\text{Gr}_H)}$

$\mathcal{F} \xrightarrow{\sigma} \mathcal{F} \ast \sigma \mathcal{F} \ast \ldots \ast \sigma^{p-1} \mathcal{F}$

\underline{Thm} \quad \underline{restrict}$\Rightarrow \text{Late-Parity}^0 \subseteq \text{Res} \Rightarrow D_{H(\mathcal{O} \times \sigma)(\text{Gr}_H)}/\text{Perf}$

\underline{Thm} \quad \underline{left}$\Rightarrow \text{Parity}^0_{(\text{Gr}_{G_H})} \subseteq P_{H(\mathcal{O})(\text{Gr}_H)}$
This is interesting even for $H = \text{GL}(1)$, for which $\text{Gr}_H$ is (étale homotopic to) a discrete union of points.
\[ P \text{ good} \]

\[ \text{Mackey-Ben} \]

\[ \text{Parey} (\hat{a}_\omega) \subset P_{\text{r.e.}} (\hat{a}_\omega) \]

\[ \text{Tilt} (\hat{a}) \subset \text{Rep} (\hat{a}) \]

**Fact:** There are enough Tilting modules
Am's k char 0 Party Perverse