Supersingular Loci of Some Unitary Shimura Varieties

Maria Fox, University of Oregon

Objectives

- Give examples of unitary Shimura varieties and the structure of their supersingular loci
- Consider their associated Rapoport-Zink spaces
- Discuss some elements used in studying the geometry of these Rapoport-Zink spaces

Historical Motivation

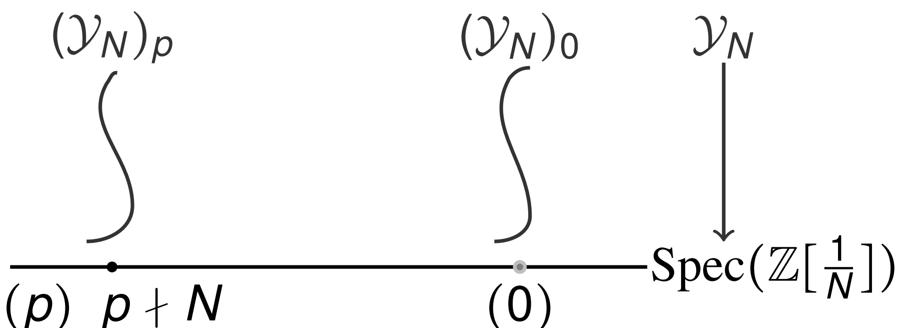
Study $\mathcal{Y}_N(\mathbb{C}) =$

$\{(E, P, Q) \mid E$ an elliptic curve $/\mathbb{C}$, (P, Q) generate $E[N](\mathbb{C})\}/\cong$

$$\coprod_{i=1}^{\varphi(N)} \Gamma(N)\backslash\mathcal{H} \xrightarrow{\sim} \mathcal{Y}_N(\mathbb{C})$$

More generally, if $N \geq 3$, $\mathcal{Y}_N$: Schms S / $\mathbb{Z}[1/N] \to$ Sets

(E, P, Q) in char p.



$$(\mathcal{Y}_N)_0(\mathbb{C}) \cong \coprod_{i=1}^{\varphi(N)} \Gamma(N)\backslash\mathcal{H}$$

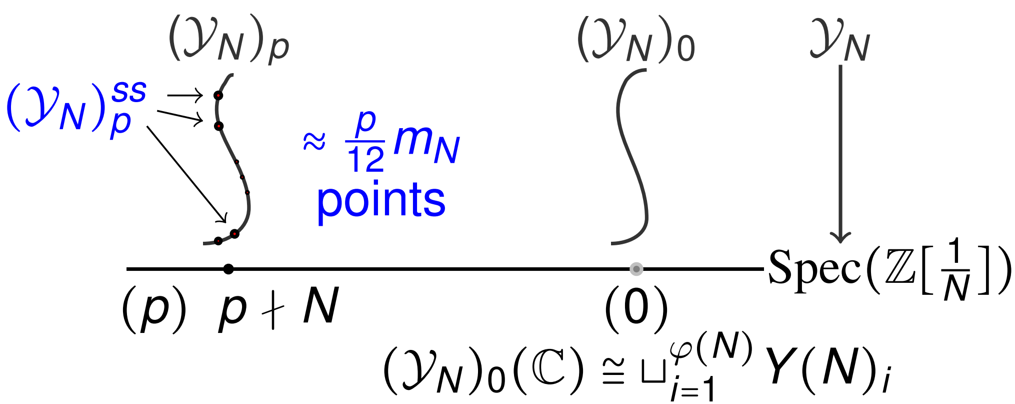
Historial Motvation

$E / \overline{\mathbb{F}}_p$ can be s.s. or ord.

The supersingular locus $(\mathcal{Y}_N)_p^{ss}$ parametrizes $(E, P, Q) \in (\mathcal{Y}_N)_p(\overline{\mathbb{F}}_p)$ where E is supersingular.

Thm (Eichler-Deuring Mass Formula): If n_p is the number of supersingular elliptic curves over $\overline{\mathbb{F}}_p$,

$$n_p \approx \frac{p}{12}$$



$\mathrm{GU}(a,b)$ Shimura Variety

$\mathcal{O}$ = int closure of $\mathbb{Z}_{(p)} \subseteq K$

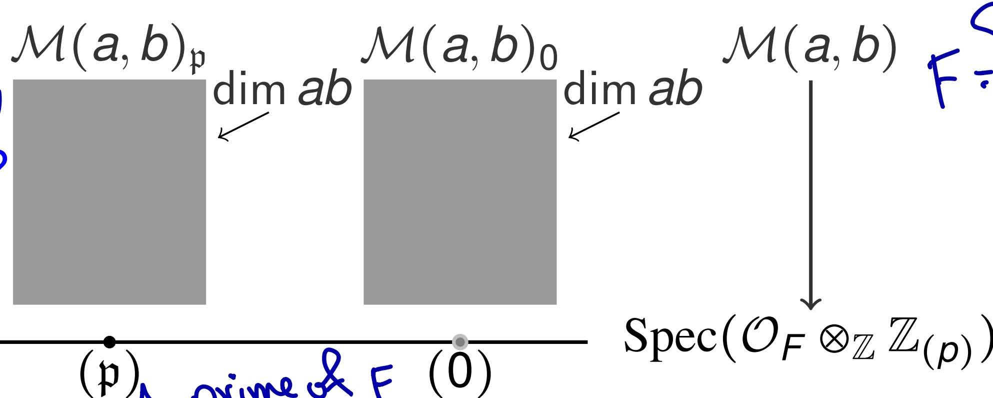
Given a quad. im. field K and $p \neq 2$ unramified in K, the $\mathrm{GU}(a,b)$ Shimura variety $\mathcal{M}(a,b)$ parametrizes $(A, \iota, \lambda, \eta)$:

- A an A.V. of dim a+b
- $\lambda : A \to A^\vee$
- ι an action of $\mathcal{O} \subseteq K$
- η level structure (hyp. at p)

$$\det(T - \iota(k); \mathrm{Lie}(A)) = (T - \varphi_1(k))^a (T - \varphi_2(k))^b.$$

mod space $(A, \iota, \lambda, \eta)$

$\mathcal{M}(a,b)^{ss}_{\mathfrak{p}}$?



$F = \begin{cases} \mathbb{Q} & a = b \\ K & a \neq b \end{cases}$

prime of F over p

The supersingular locus $\mathcal{M}(a,b)^{ss}_{\mathfrak{p}}$ parametrizes $(A, \iota, \lambda, \eta)$ where A is supersingular.

History

There are two important invariants of $\mathcal{M}(a,b)^{ss}_{\mathfrak{p}}$: the signature (a,b) and the factorization of p in K (split, inert).

- Note: as $\mathcal{M}(a,b)$ has relative dimension ab, consider $b \geq a > 0$. Also, if p splits in K, $\mathcal{M}(a,b)$ is only nonempty when $a = b$.
- In 2005, Vollaard described $\mathcal{M}(1,2)^{ss}_{p}$ when p is inert in K.
- In 2008, Vollaard and Wedhorn extended these results to $\mathcal{M}(1,n-1)^{ss}_{p}$ when p is inert in K. Most detailed descriptions in the case when $n = 2,3,4$.
- In 2013, Howard and Pappas described $\mathcal{M}(2,2)^{ss}_{p}$ when p is inert in K.
- In 2018, $\mathcal{M}(2,2)^{ss}_{p}$ when p is split in K.

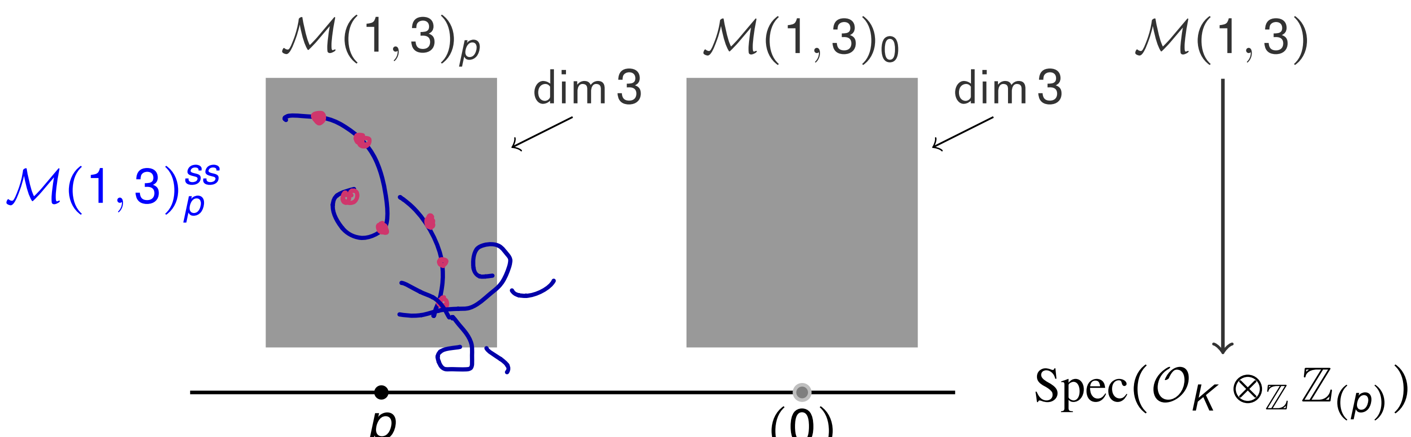
(More generally, can allow p to ramify in K. Rapoport-Terstiege-Wilson described $\mathcal{M}(1,n-1)^{ss}_{\mathfrak{p}}$ in 2013; Oki described $\mathcal{M}(2,2)^{ss}_{\mathfrak{p}}$ in 2020.)

We'll consider all nontrivial $\mathcal{M}(a,b)^{ss}_{\mathfrak{p}}$ where $a + b = 4$.

The Supersingular Locus of $\mathcal{M}(1,3)$, Inert p

Theorem (Vollaard-Wedhorn '08)

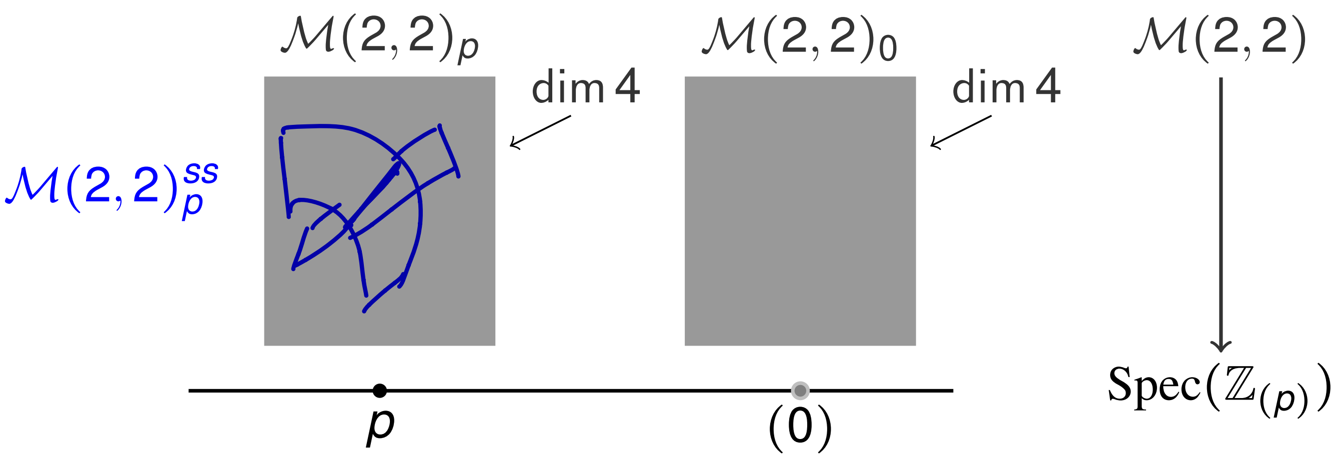
Assume that p is inert in K, and η is suff. small. The supersingular locus $\mathcal{M}(1,3)_p^{ss}$ equi-dimensional of dimension 1. Each irreducible component is isomorphic to the Fermat curve C. There are $p+1$ intersection points on each irreducible component, and each int. point is the intersection of p^3+1 irreducible components.



The Supersingular Locus of $\mathcal{M}(2,2)$, Inert p

Theorem (Howard-Pappas '13)

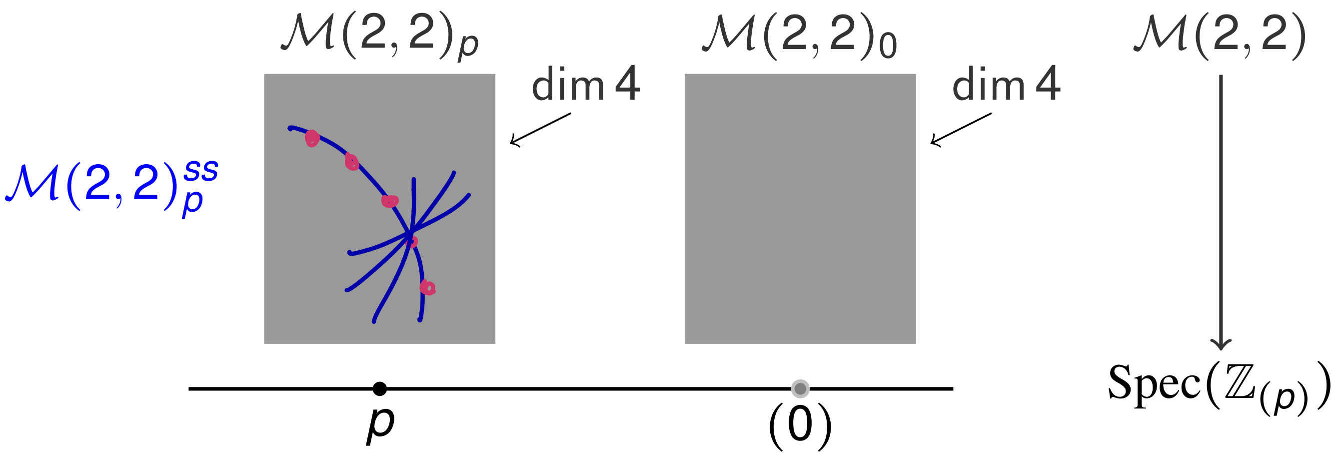
Assume that p is inert in K, and η is suff. small. The supersingular locus $\mathcal{M}(2,2)_p^{ss}$ equi-dimensional of dimension 2. Each irreducible component is isomorphic to the Fermat surface S. Any two irr. components intersect trivially, in a projective line, or in a point.



The Supersingular Locus of $\mathcal{M}(2,2)$, Split p

Theorem (F. ’18)

Assume that p is split in K, and η is suff. small. The supersingular locus $\mathcal{M}(2,2)_p^{ss}$ equi-dimensional of dimension 1. Each irreducible component is isomorphic to $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$. There are p^2+1 intersection points on each irreducible component, and each int. point is the intersection of p^2+1 irreducible components.



GL_{2n} RZ space

Rapoport-Zink Spaces

Big, formally divided by some p^n

Let $W = W(\overline{\mathbb{F}}_p)$. For $S \in \mathrm{Nilp}(W)$, let $S_0 = S \times_W \overline{\mathbb{F}}_p$.
Define: $\mathcal{N}_{2n} : \mathrm{Nilp}_W \to \mathrm{Sets}$, $\mathcal{N}_{2n}(S) = \{(G, \rho)\}/\cong$, where

- G is a supersingular p-divisible group over S of dim. n
- $\rho : G_{S_0} \to \mathbb{G}_{S_0}$ is a quasi-isogeny to a fixed basepoint p-divisible group $\mathbb{G}$ (def. over $\overline{\mathbb{F}}_p$.)

This is represented by a formal scheme, let $\mathcal{N}_{2n}$ be the underlying reduced scheme.

Example (Lubin-Tate):

$$\mathcal{N}_2 \cong \coprod_{i \in \mathbb{Z}} \mathcal{N}_2^i \cong \coprod_{i \in \mathbb{Z}} \mathrm{Spec}(\overline{\mathbb{F}}_p)$$

size of kernel

Where $\mathcal{N}_2^i$ is the locus where ρ has height i

Unitary Rapoport-Zink Spaces

$\mathcal{N}(a,b)(S) = \{(G,\iota,\lambda,\rho)\}/\cong$, where: $\mathcal{O} \subseteq K$

- G a supersingular p-div. gp over S of dim $a+b$
- $\lambda : G \xrightarrow{\sim} G^\vee$
- $\iota : \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}(G)$ of sign. (a,b)
- $\rho : G_{S_0} \to \mathbb{G}_{S_0}$, quasi-isog

Rapoport-Zink Uniformization

$$\mathcal{M}(a,b)^{ss}_{\mathfrak{p}} \cong \coprod_{j=1}^{m} \Gamma_j \backslash \mathcal{N}(a,b)$$

The Γ_j are discrete groups (depending on level structure) acting on $\mathcal{N}(a,b)$.

Can study the Rapoport-Zink spaces $\mathcal{N}(a,b)$ to understand the supersingular loci $\mathcal{M}(a,b)^{ss}_{\mathfrak{p}}$

The $\mathrm{GU}(1,3)$ Rapoport-Zink Space

$\mathcal{N}^{4i}_{(1,3)}$ locus on which ρ has height $4i$

Thm (Vollaard-Wedhorn):

For p inert in K

- The $\mathrm{GU}(1,3)$ Rapoport-Zink space $\mathcal{N}(1,3)$ decomposes into connected components as $\mathcal{N}(1,3) = \bigsqcup_{i \in \mathbb{Z}} \mathcal{N}^{4i}_{(1,3)}$.
- Each irr. comp. of $\mathcal{N}^{4i}_{(1,3)}$ is isom to:
$$C \;:\; x_0^{p+1} + x_1^{p+1} + x_2^{p+1} = 0 \subset \mathbb{P}^2_{\overline{\mathbb{F}}_p}.$$
- These irr. comp. are indexed by vertex lattices Λ of type 3, inside a fixed hermitian space W.
- Each irr. comp. contains $p^3 + 1$ int. pts, and each int pt is the intersection of $p + 1$ irr. comp.
- $\mathcal{N}(1,3)$ has two Ekedahl-Oort strata: the superspecial points, which are precisely the int. points, and their complement.

Deligne-Lusztig Varieties

- Given G over $\mathbb{F}_q$ a reductive group (with Frob. F), $B \subset G(\overline{\mathbb{F}}_q)$ an F-stable Borel, and $w \in W$:

$$X(w) = \{gB \in G/B \mid g^{-1}F(g) \in BwB\}.$$

- Can think of $X(w)$ as a moduli space of Borel subgroups B such that B' and $F(B')$ have "relative position w"
- Example: $G = \mathrm{SL}_2$ over $\mathbb{F}_q$, B upper-triangular Borel. Note that $G/B \leftrightarrow \mathbb{P}^1_{\overline{\mathbb{F}}_q}$, and that $W = \{1, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\}$.

$gB \mapsto g \cdot e_1$

- Two lines in $\overline{\mathbb{F}}_q^2$ have rel. pos. 1 iff they are equal.
- So,

$$X(1) = \mathbb{P}^1(\mathbb{F}_q)$$

$$X\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q).$$

- The Fermat curve C arises as a DL variety for a unitary group G and parabolic subgroup $P \subset G$.

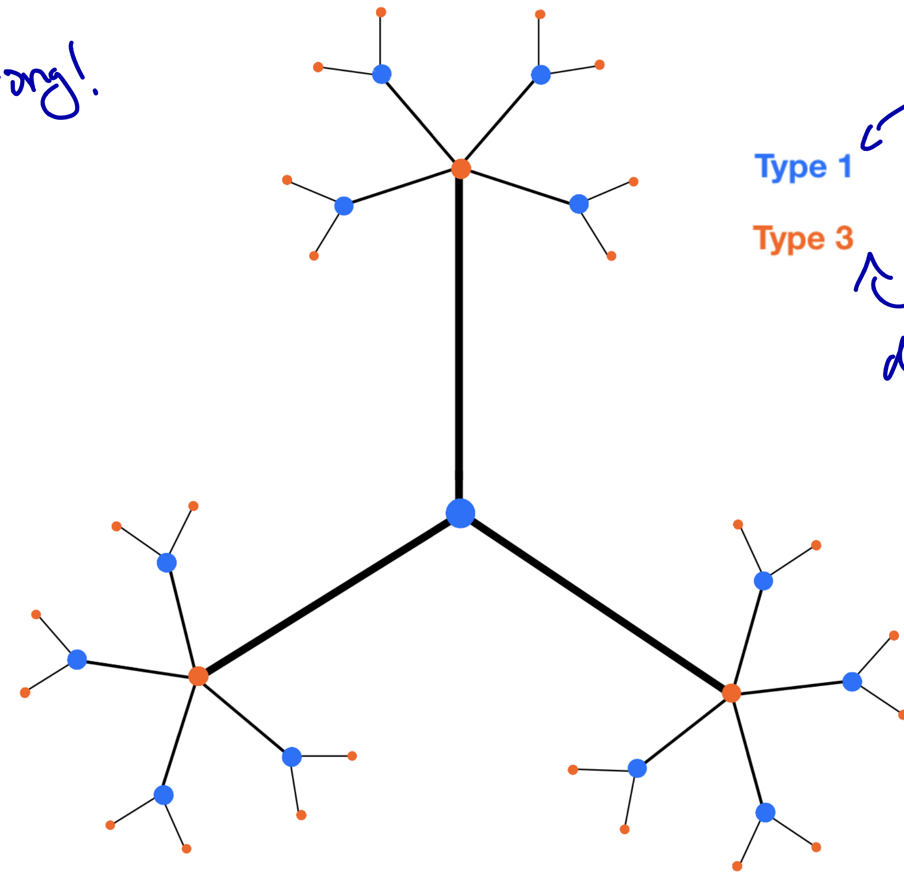
(1,n-1)

Bruhat-Tits Building

- From the Hermitian space W over $\mathbb{Q}_p$ of Vollaard-Wedhorn, can construct a simplicial complex $\mathcal{B}$:
- The 0-cells are given by vertex lattices: $\mathbb{Z}_p$-lattices $\Lambda \subset W$ such that $p\Lambda \subset \Lambda^\vee \subset \Lambda$ dual lattice (up to norm.)
- There is an m-simplex connecting $\Lambda_0, \Lambda_1, \ldots, \Lambda_m$ whenever $\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_m$
- $\mathcal{B}$ is isom. to the Bruhat-Tits building of $\mathrm{SU}(W)$.
- For sign. (1,3): $\mathcal{B}$ is a tree, with two types of vertices: "Type-1" and "Type-3."
  Type-1 vertices have degree $p+1$, Type-3 have p^3+1.
  Type-1 vertices are only adjacent to Type-3 and vice versa.
- These combinatorics are reflected in the intersection combinatorics of $\mathcal{N}(1,3)$.

$\mathcal{B}$ Sign(1,3)

Slightly wrong!



Type 1

should have deg $p+1$

Type 3

should have degree p^3+1

Ekedahl-Oort Strata

- The Ekedahl-Oort stratification of $\mathcal{N}(a,b)$ is based on the isom class. of the *p*-torsion subgroups
- That is: two points (G,ι,λ,ρ) and $(G',\iota',\lambda',\rho')$ of $\mathcal{N}(a,b)(\overline{\mathbb{F}}_p)$ are in the same EO stratum if and only if $G[p] \cong G'[p]$.
- A point $(G,\iota,\lambda,\rho) \in \mathcal{N}(a,b)(\overline{\mathbb{F}}_p)$ is called superspecial if and only if G is isomorphic to (not just isog. to!) the p-div gp of a product of supersingular elliptic curves.
- In the examples today, the superspecial points form the 0-dimensional Ekedahl-Oort stratum.

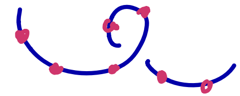
The $\mathrm{GU}(1,3)$ Rapoport-Zink Space Again

Thm (Vollaard-Wedhorn):

For p inert in K

- The $\mathrm{GU}(1,3)$ Rapoport-Zink space $\mathcal{N}(1,3)$ decomposes into connected components as $\mathcal{N}(1,3) = \bigsqcup_{i \in \mathbb{Z}} \mathcal{N}^{4i}_{(1,3)}$.
- Each irr. comp. of $\mathcal{N}^{4i}_{(1,3)}$ is isom to:
$$C : x_0^{p+1} + x_1^{p+1} + x_2^{p+1} = 0 \subset \mathbb{P}^2_{\overline{\mathbb{F}}_p}.$$
- These irr. comp. are indexed by vertex lattices Λ of type 3, inside a fixed hermitian space W.
- Each irr. comp. contains $p^3 + 1$ int. pts, and each int pt is the intersection of $p + 1$ irr. comp.
- $\mathcal{N}(1,3)$ has two Ekedahl-Oort strata: the superspecial points, which are precisely the int. points, and their complement.

The $\mathrm{GU}(1,3)$ Rapoport-Zink Space Again



For p inert in K:

Thm (Vollaard-Wedhorn), Rephrased:

- The $\mathrm{GU}(1,3)$ Rapoport-Zink space $\mathcal{N}(1,3)$ decomposes into connected components based on height of ρ.
- The irr. components of each fixed connected component are indexed by certain vertices in a Bruhat-Tits building. The intersection combinatorics reflect the combinatorics of this building.
- Each irr. component is isomorphic to a particular kind of Deligne-Lusztig variety.
- The EO strata respect this structure, the (closure of) each EO in an irr. comp. is also a Deligne-Lusztig variety.

exception b/w $GU^0(N) \subsetneq GU(N)$

The $\mathrm{GU}(2,2)$ Rapoport-Zink Space

$\cong$ an orthog gp

Thm (Howard-Pappas):

For p inert in K

- The $\mathrm{GU}(2,2)$ Rapoport-Zink space $\mathcal{N}_{(2,2)}$ decomposes into connected components as $\mathcal{N}(2,2) = \bigsqcup_{i \in \mathbb{Z}} \mathcal{N}^{4i}_{(2,2)}$.
- Each irr. comp. of $\mathcal{N}^{4i}(2,2)$ is isom to:

$$S \;:\; x_0^{p+1} + x_1^{p+1} + x_2^{p+1} + x_3^{p+1} = 0 \subset \mathbb{P}^3_{\overline{\mathbb{F}}_p}.$$

also a DL variety

- These irr. comp. are indexed by vertex lattices Λ of type 6, inside a fixed quadratic space W. 2, 4, 6
- Any two irr. comp. intersect trivially, intersect in a single point, or have intersection isomorphic to C. These int. combinatorics are controlled by the BT building of $\mathrm{SO}(W)$.
- The superspecial points (minimal EO stratum) are exactly these intersection points.

The $\mathrm{GU}(2,2)$ Rapoport-Zink Space

(G, i, λ, ρ)

$\rho : G \to G$

$\rho : G_0 \times G_1 \to G_0 \times G_1$

$\rho_0 : G_0 \to G_0$

Thm (F.):

For p split in K

- The $\mathrm{GU}(2,2)$ Rapoport-Zink space $\mathcal{N}(2,2)$ decomposes into connected components as $\mathcal{N}(2,2) = \bigsqcup_{i \in \mathbb{Z}, j \in \mathbb{Z}} \mathcal{N}^{4i}_{j,(2,2)}$. height of ρ_0
- Each irr. comp. of $\mathcal{N}^{4i}_{j,(2,2)}$ is isom to $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$. DL variety
- These irr. comp. are indexed by vertex lattices Λ of type 4, inside a fixed quadratic space W. 2, 4
- Each irr. comp. contains $p^2 + 1$ int. pts, and each int pt is the intersection of $p^2 + 1$ irr. comp. BT building $SO(W)$
- $\mathcal{N}(2,2)$ has two EO strata: the superspecial points, which are precisely the int. points, and their complement.

General Situation

- The Rapoport-Zink spaces $\mathcal{N}(1, n-1)$ (p inert), $\mathcal{N}(2,2)$ (p inert), and $\mathcal{N}(2,2)$ (p split) have similar structure: their irr. components are Deligne-Lusztig varieties and intersection combinatorics coming from a Bruhat-Tits building. The Ekedahl-Oort stratification respects decomp. into Deligne-Lusztig varieties.

$\mathcal{N}(1,n-1)$

- These are all examples of Rapoport-Zink spaces of Coxeter Type. Görtz, He, and Nie have classified which Rapoport-Zink spaces have this structure, after perfection. The first paper (2013) lists all 21 possibilities.
- There are also examples of Rapoport-Zink spaces that do not have this structure: for example, those coming from Siegel modular varieties for $g \geq 3$ and some unitary Rapoport-Zink spaces.

GL_4

A Particular Rapoport-Zink Space

$\mathcal{N}_4(\overline{\mathbb{F}}_p) = \{(G,\rho)\}/\cong$, where G is of dim 2 s.s.

$\mathcal{O} \subseteq K$

$\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \subseteq K \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathbb{Q}_p \times \mathbb{Q}_p$

$\mathcal{N}(2,2)(\overline{\mathbb{F}}_p) = \{(G,\iota,\lambda,\rho)\}/\cong$, where:

- G a p-divisible gp of dim 4 s.s.
- $\lambda : G \xrightarrow{\sim} G^\vee$
- $\iota : \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}(G)$ of sign. $(2,2)$
- $\rho : G \to \mathbb{G}$, quasi-isog

$G \cong G_0 \times G_1 \longrightarrow \mathbb{G}_0 \times \mathbb{G}_1$

$\rho : G_0 \to \mathbb{G}_0$

When p splits in K

Isomorphism:

ht $\rho = 0$

$$\varphi : \mathcal{N}^0_{(2,2)} \to \mathcal{N}_4$$

s.s. of dim 2

$$(G,\iota,\lambda,\rho) \mapsto (G_0, \rho|_{G_0} : G_0 \to \mathbb{G}_0)$$

With inverse:

$$\varphi^{-1} : \mathcal{N}_4 \to \mathcal{N}^0_{(2,2)}$$

ht 0

$$(G_0,\rho_0) \mapsto (G_0 \times G_0^\vee, \iota, \lambda, \rho_0 \times (\rho_0^\vee)^{-1})$$

Structure of $\mathcal{N}_4$

Thm (F.):

- The GL_4 Rapoport-Zink space $\mathcal{N}_4$ decomposes into connected components as $\mathcal{N}_4 = \bigsqcup_{i\in\mathbb{Z}} \mathcal{N}_4^i$. ← "ht ρ_0"
- Each irr. comp. of $\mathcal{N}_4^i$ is isom to $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$. } DL
- These irr. comp. are indexed by vertex lattices Λ of type 4, inside a fixed quadratic space W.
- Each irr. comp. contains p^2+1 int. pts, and each int pt is the intersection of p^2+1 irr. comp.

BT building

- $\mathcal{N}_4$ has two EO strata: the superspecial points, which are precisely the int. points, and their complement.

(When p split, this description plus the observation $\mathcal{N}^0_{(2,2)} \cong \mathcal{N}_4$ yields the description of $\mathcal{N}(2,2)$.)

$V/W(\overline{\mathbb{F}}_p)_{\mathbb{Q}}$

Quadratic Space and Building

- Let $\mathbb{G}$ be the basepoint for $\mathcal{N}_4$. Construct $W = V^{\Phi}$, a $\mathbb{Q}_p$-vector space of special quasi-endomorphisms of $\mathbb{G} \times \mathbb{G}^{\vee}$, with quadratic composition form.

dual lattice

- A vertex lattice is a $\mathbb{Z}_p$-lattice $\Lambda \subset V^{\Phi}$ such that $p\Lambda \subset \Lambda^{\vee} \subset \Lambda$. The type of Λ is $\dim_{\mathbb{F}_p}(\Lambda/\Lambda^{\vee})$. Based on V^{Φ}, the type can only be 2 or 4.
- The BT Building of $\mathrm{SO}(V^{\Phi})$ is a tree, with vertices formed by vertex lattices. There is an edge between Λ and Λ' iff one is contained in the other.
- Type-2 vertices are only adjacent to type-4 vertices, and vice versa. Both types of vertices have degree $p^2 + 1$.

This building relates to $\mathcal{N}_4$ in two ways:

1. Given Λ of type 4, should have irr. comp. $\mathcal{N}_{\Lambda}^{0}$, a DL variety.
2. Intersection combinatorics of the $\mathcal{N}_{\Lambda}^{0}$ should relate to combinatorics of the tree.

V^Φ = special quasi end of $G\times G^\vee$

$\rho: G\to G$

V = special quasi end of $N\times N^\vee$

Irreducible Component Indexed by Λ

$\rho: G\times G^\vee \to G\times G^\vee$

pht 0 ← need for irr. comp

- Given $\Lambda \subset V^\Phi$ of type d (2 or 4), $\subseteq V^\Phi \subseteq V$

proj variety

$$\mathcal{N}^0_\Lambda = \{(G,\rho)\in\mathcal{N}^0_4 \mid \rho^*(v)\in \mathrm{End}(G\times G^\vee)\ \forall v\in\Lambda^\vee\}$$

Require $\rho^*(v)$ is a true endmorphism, not just quasi-end.

- Can use Dieudonné theory (etc) to identify:

linear alg on M

$\rho: G\to G$ ← crystal G (v space of 4)

$M/W(\overline{\mathbb{F}}_p)$

$$\mathcal{N}^0_\Lambda(\overline{\mathbb{F}}_p) = \{M\subset \mathbb{N} \mid M \text{ is a Dieudonné module, } \Lambda^\vee \text{ acts correctly}\}$$

$pM \subseteq F^{-1}(pM) \subseteq M$

the lattice of $v\in V$ that are "special end for M"

$\Lambda^\vee \subseteq L \subseteq \Lambda$

$$= \{L\subset V \mid L \text{ “a special end. lattice,” and } \Lambda^\vee \subset L\}$$

$\mathcal{L} \subseteq (\Lambda/\Lambda^\vee)_{\overline{\mathbb{F}}_p}$

dmd

$$= \{\mathcal{L}\in \mathrm{OGr}(\Lambda/\Lambda^\vee)(\tfrac{d}{2})(\overline{\mathbb{F}}_p) \mid \dim(\mathrm{Frob}(\mathcal{L})\cap\mathcal{L}) = \tfrac{d}{2}-1\}^+$$

subspaces w/ fixed relative position to Frob-twist

This is a Deligne-Lusztig variety for $\mathrm{SO}(\Lambda/\Lambda^\vee)$.

- If $d = \cancel{1}\ 2$, $\mathcal{N}^0_\Lambda$ is a single point. If $d = \cancel{2}\ 4$, $\mathcal{N}^0_\Lambda \cong \mathbb{P}^1$.

Λ_4's give irr comp $\cong \mathbb{P}^1$ Λ_2's give intersection pts $\cong \{\cdot\}$

Combinatorics from Building

- Given $\Lambda_d \subset V^\Phi$ of type d (2 or 4),

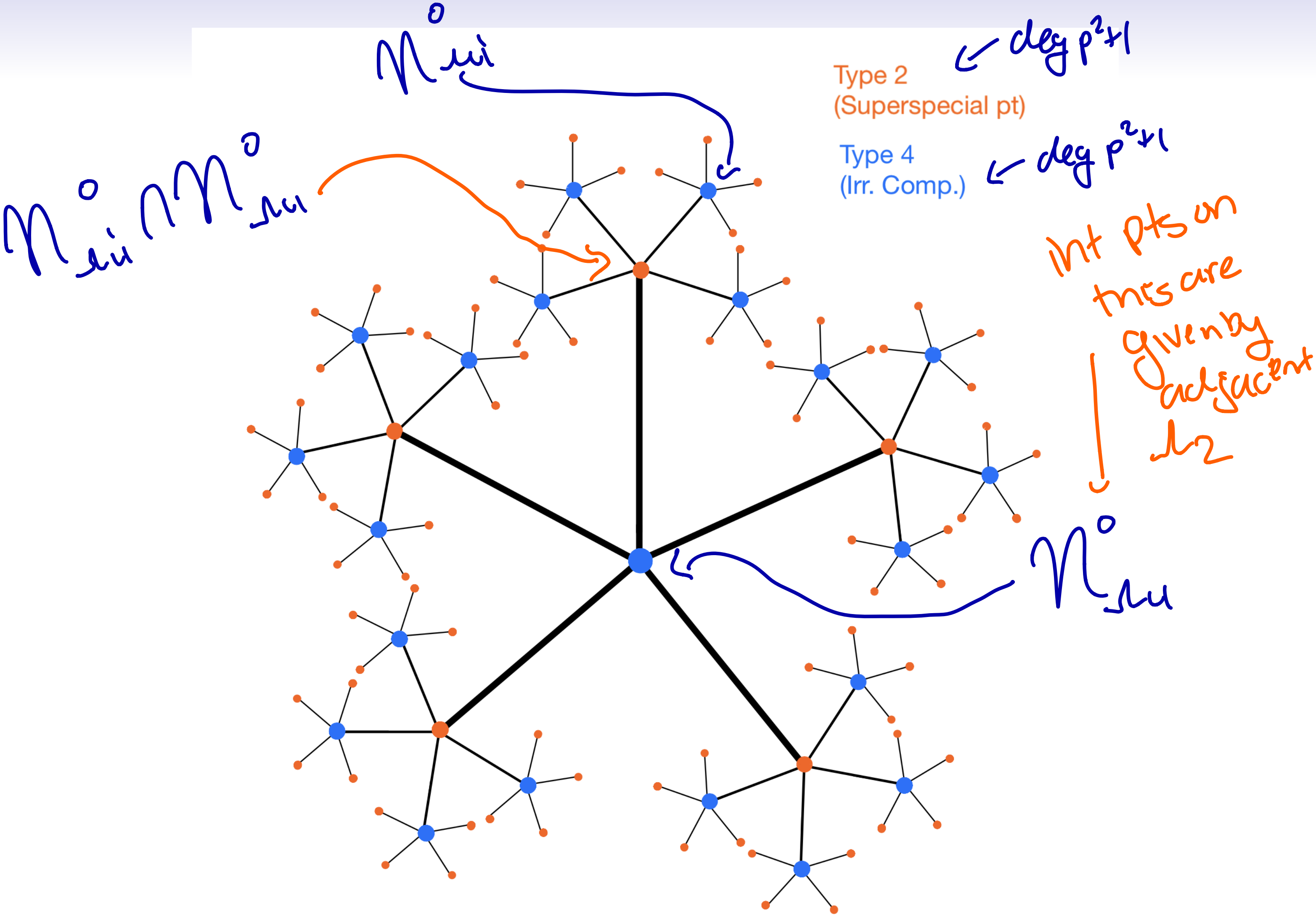
$$\mathcal{N}^0_{\Lambda_d} = \{(G, \rho) \in \mathcal{N}^0_4 \mid \rho^*(v) \in \mathrm{End}(G \times G^\vee)\ \forall v \in \Lambda_d^\vee\}$$

- Note: $\mathcal{N}^0_{\Lambda_2} \subset \mathcal{N}^0_{\Lambda_4}$ if and only if $\Lambda_2 \subset \Lambda_4$
  $\Lambda_4 \cap \Lambda_4'$ is either a type-2 lattice or trivial.
  $\mathcal{N}^0_{\Lambda_4} \cap \mathcal{N}^0_{\Lambda_4'}$ is either $\mathcal{N}^0_{\Lambda_4 \cap \Lambda_4'}$ or empty.

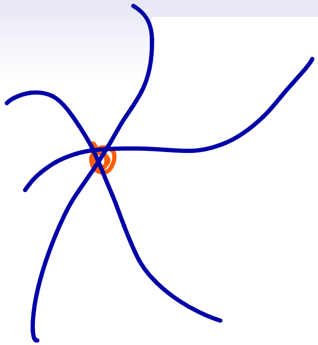
$\Lambda_2 \subseteq \Lambda_4$ $\Lambda_4^\vee \subseteq \Lambda_2^\vee$
(G, ρ) is "integral wrt $\Lambda_2^\vee$"
$\Rightarrow (G, \rho)$ is "integral wrt $\Lambda_4^\vee$"

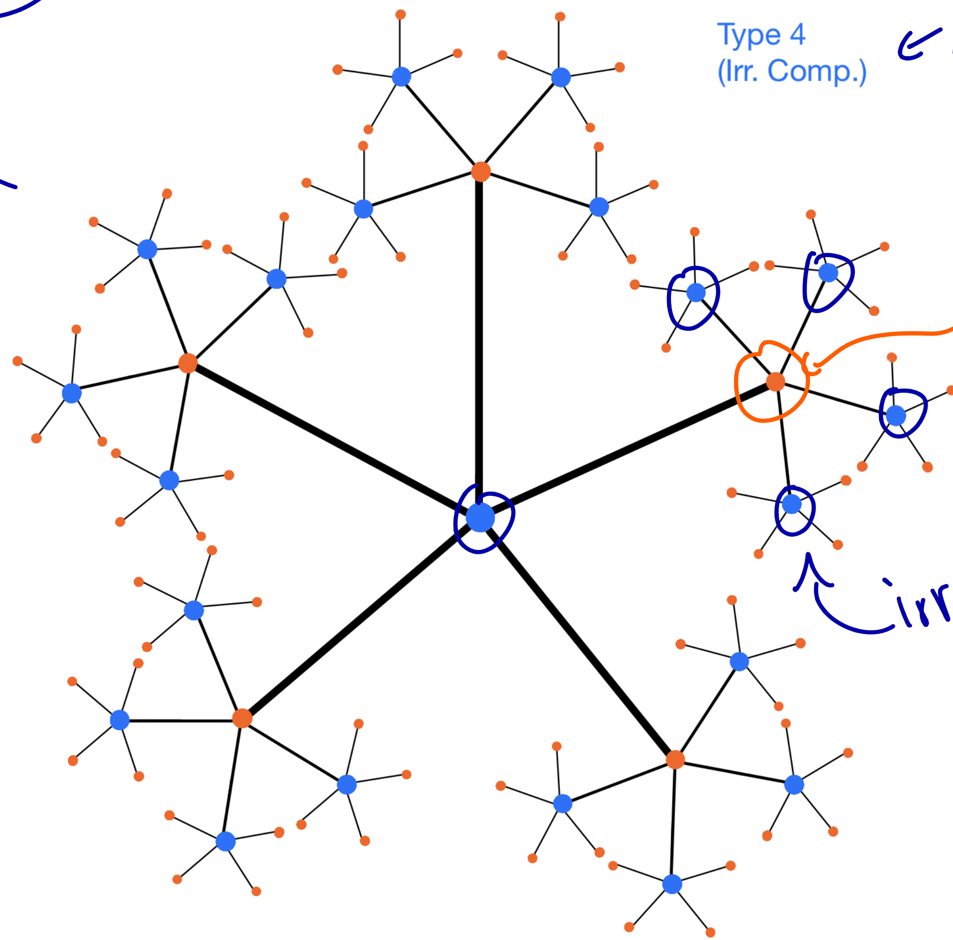
- Fix vertex lattice Λ_4, irr. comp $\mathcal{N}^0_{\Lambda_4}$. The int. pts on $\mathcal{N}^0_{\Lambda_4}$ are exactly the $\mathcal{N}^0_{\Lambda_2} \subset \mathcal{N}^0_{\Lambda_4}$, param. by $\Lambda_2 \subset \Lambda_4$. Since $v(\Lambda_4)$ has degree $p^2 + 1$, there are $p^2 + 1$ such int. pts.

- Fix vertex lattice Λ_2, pt $\mathcal{N}^0_{\Lambda_2}$. The irr. comps $\mathcal{N}^0_{\Lambda_4}$ containing $\mathcal{N}^0_{\Lambda_2}$ are given by Λ_4 containing Λ_2. Since $v(\Lambda_2)$ has degree $p^2 + 1$, there are $p^2 + 1$ such irr. comps.



Type 2
(Superspecial pt)
Type 4
(Irr. Comp.)
deg p^2+1
deg p^2+1
Int pts on these are given by adjacent Λ_2





Type 2 (Superspecial pt) $\leftarrow$ deg p^2+1

Type 4 (Irr. Comp.) $\leftarrow$ deg p^2+1

$\mathcal{M}^{\circ}_{\mathcal{A}_2}$ superspecial pt

$\uparrow$ irr comps passing through that pt

Some Takeaways

1. In some cases, the supersingular loci $\mathcal{M}(a,b)^{ss}_{\mathfrak{p}}$ have especially nice structure (can be written as a union of Deligne-Lusztig varieties, etc.)
2. The Rapoport-Zink spaces $\mathcal{N}(a,b)$ occur naturally in the study of $\mathcal{M}(a,b)^{ss}_{\mathfrak{p}}$. There is also some especially nice structure (role of Bruhat-Tits building) that is more visible on $\mathcal{N}(a,b)$.
3. Warning! This does not hold in general for all Shimura varieties of PEL-type, or even for all unitary Shimura varieties.

$\mathcal{N} \leftarrow GU_4$ or $GU(2,2)$

$\mathcal{N}_v = \{ (X, \rho) \mid$ single $v \in V_\Phi$ is integral $\}$

Thank you!