Hodge--Tate decomposition for non-smooth rigid spaces

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Complex theory.

Let X be a sm proper var / $\mathbb{C}$

$$X \rightsquigarrow X^{an} \rightsquigarrow H^*_{Sing}(X^{an}, \mathbb{C})$$

$$H^*_{dR}(X/\mathbb{C}) = H^*(X, \Omega^\bullet_{X/\mathbb{C}}).$$

a Hodge filtration.

$$E_1^{pq} = H^q(X, \Omega^p_{X/\mathbb{C}}) \Rightarrow H^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}).$$

Now, if X is proper / $\mathbb{C}$.

$$H^*_{Sing}(X^{an}, \mathbb{C}) \cong H^*(X, \underline{\Omega}^\bullet_{X/\mathbb{C}})$$

Deligne - Du Bois cpx.

$\underline{\Omega}_{X/\mathbb{C}}$: a simplicial resolution of singularities $P: X_\bullet \to X$, $f_n: X_n \to X$

$$\underline{\Omega}^\bullet_{X/\mathbb{C}} = RP_* \Omega^\bullet_{X_\bullet/\mathbb{C}}$$
$$= \varprojlim_{[n]} Rf_{n*} \Omega^\bullet_{X_n/\mathbb{C}} \qquad \Omega^{\geq i}_{X_n/\mathbb{C}}, \ \forall n.$$

$$(X_\bullet \to X, \quad (\cdots \substack{\to\\\to\\\to} X_1 \rightrightarrows X_0) \to X \qquad \in D_{coh}(X))$$

$\rightsquigarrow$ Hodge filtration on $\underline{\Omega}^\bullet_{X/\mathbb{C}}$, $\quad gr^i \underline{\Omega}^\bullet_{X/\mathbb{C}} =: \underline{\Omega}^i_{X/\mathbb{C}}[-i]$

$\rightsquigarrow$ Hodge filtration on $H^*_{Sing}(X^{an}, \mathbb{C})$

Singular version Hodge - dR also degenerates at E_1

p-adic theory. (K finite ext of $\mathbb{Q}_p$).

Rigid spaces. : p-adic analogue of complex analytic spaces

eg. algebraic varieties (analytification)

p-adic formal scheme over $\mathbb{Z}_p$ $\rightsquigarrow$ generic fibers.
$\widehat{(f.\ .f.t)}$

Drinfeld upper plane.

$H^*_{ét}(X_{\bar{K}}, \mathbb{Q}_p)$. p-adic analogue of singular coh.
$\circlearrowleft$
$Gal(\bar{K}/K)$.

Theorem. (Faltings, Scholze)

X sm proper rigid spaces / K.

$$H^n_{ét}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \otimes_K \mathbb{C}_p(-j).$$

, Galois equivariant.

Note.

This builds a connection between arithmetic information & geometric information.

Theorem (Tate).

$$H^0(Gal(K), \mathbb{C}_p(j)) = \begin{cases} 0, & j \neq 0 \\ K. & j = 0 \end{cases}.$$

Theorem (G.)

X proper rigid space $/K$.

$$H^n_{ét}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i+j=n} \underbrace{H^i(X, \underline{\Omega}^j_{X/K})}_{\text{trivial Galois action}} \otimes_K \mathbb{C}_p(-j)$$

Galois equivariant.

· RMK.

$\underline{\Omega}^j_{X/K}$ is constructed the same way as the schematic version (Temkin).

· RMK.

· For algebraic varieties, it's proved by Beilinson.

Before that, Kisin uses the alteration to show p-adic étale coh & de Rham.

· RMK.

There is also a version of de Rham comparison.

· RMK.

There is a version for rigid spaces $/\mathbb{C}_p$, not necessarily defined over K.

X proper $/\mathbb{C}_p$,
$$H^n_{ét}(X, \mathbb{Q}_p) \otimes \mathbb{C}_p = \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{C}_p})$$

Not canonical, not Galois equivariant.

· Sketch of proof.

Primitive comparison (Scholze)

X proper $/\mathbb{C}_p$, $\quad H^*_{ét}(X, \mathbb{Q}_p) \otimes \mathbb{C}_p \cong H^*_{proét}(X, \hat{\mathcal{O}}_X)$

$\nu : X_{proét} \longrightarrow X_{ét}$

$\leadsto$ breaks into two steps : ① $R\nu_* \hat{\mathcal{O}}_X$

② $H^*_{ét}(X, R\nu_* \hat{\mathcal{O}}_X)$

Theorem 13 (G.)

X proper $/\mathbb{C}_p$, dim n.

(1) $R\nu_* \hat{\mathcal{O}}_X$ admits a natural ascending filtration, s.t. graded piece
$$gr^i(R\nu_* \hat{\mathcal{O}}_X) = \Omega^i_X \quad [-i]$$

(2) If X is defined over K, then the filtration degenerates canonically in the derived category.

$$R\nu_*\hat{\mathcal{O}}_X \cong \bigoplus_i \underline{\Omega}^i_{X/K}(-i)[-i]$$

(3) ~~If X~~ In general, the filtration degenerates non-canonically.

(there is a qis).

$$R\nu_*\hat{\mathcal{O}}_X \cong \bigoplus \underline{\Omega}^i_{X/\mathbb{C}_p}[-i]$$

This implies HT decomposition.

$$\begin{aligned} H^n_{et}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p &\cong H^n_{proet}(X, \hat{\mathcal{O}}_X) \\ &= H^n_{et}(X, R\nu_*\hat{\mathcal{O}}_X) \\ &\cong H^n_{et}(X, \bigoplus \underline{\Omega}^i_{X/\mathbb{C}_p}[-i]) \\ &= \bigoplus H^{n-i}_{et}(X, \underline{\Omega}^i_{X/\mathbb{C}_p}). \end{aligned}$$

(for X defined over K. $H^{n-i}(X, \underline{\Omega}^i_{X/K}) \otimes_K \mathbb{C}_p(-i)$

Application.

Theorem (Bhatt – Scholze).

X rigid space of dim n.

$R\nu_*\hat{\mathcal{O}}_X \in D^{[0,n]}(X)$.

So combine with the Theorem above

$\Rightarrow$ when X proper, $R\nu_*\hat{\mathcal{O}}_X \cong \bigoplus \underline{\Omega}^i_{X/\mathbb{C}_p}[-i]$.

$\Rightarrow$ $\underline{\Omega}^i_{X/\mathbb{C}_p} \in D^{[0,n-i]}(X)$.

(Guillen – Navarro , Aznar – Puerta – Steenbrink).

classical proof uses MHS. (Peters – Steenbrink)

Sketch of Theorem B.

Main idea: Deligne – Illusie.

splitting of $R\nu_*\hat{\mathcal{O}}_X$ $\Leftarrow$ lifting of rigid space along $B^+_{dR}/\xi^2 \to \mathbb{C}_p$.

First assume X smooth rigid space.

Step 1.

$$(\tau^{\leq 1} R\nu_* \hat{\mathcal{O}}_X)(1)[1] \cong \tau^{\leq 0} R\nu_* \widehat{\mathbb{L}}^{\cdot}_{\hat{\mathcal{O}}_X/A_{\inf}[\frac{1}{p}]} \cong \widehat{\mathbb{L}}_{\mathcal{O}_X/A_{\inf}[\frac{1}{p}]}$$

Step 2. deformation of rigid spaces.

$$\widehat{\mathbb{L}}_{\mathcal{O}_X/A_{\inf}[\frac{1}{p}]} \cong \tau^{\geq -1} \widehat{\mathbb{L}}^{\cdot}_{\mathcal{O}_X/(A_{\inf}[\frac{1}{p}]/\xi^2)}$$

B_{dR}^+/ξ^2.

if X admits a flat lift to B_{dR}^+/ξ^2, then RHS splits in the derived? category.

(if X is defined over K, $\Rightarrow$ get a canonical lift via $K \hookrightarrow B_{dR}^+/\xi^2$
if X is proper over $\mathbb{C}_p$ $\Rightarrow$ get a lift. (Conrad - Gabber)

assume those two $\Rightarrow$ $\tau^{\geq -1} \widehat{\mathbb{L}}_{\mathcal{O}_X/A_{\inf}[\frac{1}{p}]/\xi} \cong \mathcal{O}_X(1)[+1] \oplus \Omega^1_X$

Step 3.

$$\Rightarrow \mathcal{O}_X \oplus \Omega^1_X(-1)[-1] \longrightarrow \tau^{\leq 1} R\nu_* \hat{\mathcal{O}}_X$$

by Deligne-Illusie

$$\Longrightarrow \bigoplus_i \Omega^i_X(-i)[-i] \longrightarrow R\nu_* \hat{\mathcal{O}}_X.$$

it's a qis, by Künneth of proét coh [BMS1].

reduce to the curve, ✓.

Step. 4.

Extend previous steps to the. m-truncated simplicial space.

The rest : simplicial approximation $\Longrightarrow$ general non smooth case