Compatibility of the Fargues-Scholze and Gan-Takeda Local Langlands Correspondences

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Notation

We let:

1. p and ℓ be distinct primes.
2. $G/\mathbb{Q}_p$ be a connected reductive group.
3. $W_{\mathbb{Q}_p}$ be the Weil group of $\mathbb{Q}_p$
4. $\hat{G}$ the Langlands dual group of G viewed as a reductive group over $\overline{\mathbb{Q}}_\ell$
5. ${}^L G := W_{\mathbb{Q}_p} \ltimes \hat{G}$
6. $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$ and $j : \overline{\mathbb{Q}}_p \xrightarrow{\simeq} \mathbb{C}$ be fixed isomorphisms.

Notation

We set:

1. $\Pi(G)$ to be isomorphism classes of smooth irreducible representations of $G(\mathbb{Q}_p)$.
2. $\Phi(G)$ to be the set of conjugacy classes of admissible homomorphisms:

$$\phi : W_{\mathbb{Q}_p} \times SL(2, \overline{\mathbb{Q}}_\ell) \to {}^LG(\overline{\mathbb{Q}}_\ell)$$

3. $\Phi^W(G)$ to be the set of conjugacy classes of continuous semisimple homomorphisms:

$$\phi : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$$

Theorem (Harris-Taylor/Henniart/Scholze)

Let $G = GL_n$ then, for every $n \geq 1$, there exists a unique bijection:

$$\Pi(G) \xrightarrow{LLC_n} \Phi(G)$$

$$\pi \mapsto \phi_\pi$$

generalizing local class field theory and characterized by the preservation of character twists, L, ϵ, and γ factors.

Theorem (Fargues-Scholze)

For any G, there exists a map:

$$\Pi(G) \xrightarrow{LLC_G^{FS}} \Phi^W(G)$$

$$\pi \mapsto \phi_\pi^{FS}$$

enjoying the following properties:

1. It is compatible with tensor product of representations.
2. It is is compatible with parabolic induction of representations.
3. It is compatible with the correspondence of Harris-Taylor/Henniart for $G = GL_n$ and its inner forms.

Question:

Can we show that the Fargues-Scholze Local Langlands correspondence is compatible with other instances of the correspondence? Namely, given a "known local Langlands correspondence":

$$\Pi(G) \xrightarrow{LLC_G} \Phi(G)$$

We expect a commutative diagram of the form:

$$\begin{tikzcd} \Pi(G) \arrow[r, "LLC_G"] \arrow[dr, "LLC_G^{FS}"'] & \Phi(G) \arrow[d] \\ & \Phi^W(G) \end{tikzcd}$$

where the right horizontal arrow precomposes the map $\phi \in \Phi(G)$ with $g \in W_{\mathbb{Q}_p} \mapsto (g, \begin{pmatrix} |g|^{\frac{1}{2}} & 0 \\ 0 & |g|^{\frac{-1}{2}} \end{pmatrix}) \in W_{\mathbb{Q}_p} \times SL(2, \overline{\mathbb{Q}}_\ell)$

Theorem (Gan-Takeda/Gan-Tantono)

1. Let $L/\mathbb{Q}_p$ be a finite extension.
2. $G = Res_{L/\mathbb{Q}_p} GSp_4$ or $G = Res_{L/\mathbb{Q}_p} GU_2(D)$, where D is the quaternion division algebra over L.
3. Up to the choice of the fixed isomorphism i, there exists a unique map:

$$LLC_G : \Pi(G) \to \Phi(G)$$

$$\pi \mapsto \{\phi_\pi : W_L \times SL(2, \overline{\mathbb{Q}}_\ell) \to \hat{G}(\overline{\mathbb{Q}}_\ell) = GSpin_5(\overline{\mathbb{Q}}_\ell) \simeq GSp_4(\overline{\mathbb{Q}}_\ell)\}$$

characterized by preservation of character twists, L, ϵ, γ factors, and a condition on the Plancharel measure of a family of induced representations.

Remarks

1. For the representations π contained in an L-packet with no supercuspidals, compatibility with the Fargues-Scholze LLC follows formally from known properties of the Fargues-Scholze correspondence.
2. We will be interested in the case where a representation π of G is in an L-packet containing only supercuspidals, which implies that ϕ_π is supercuspidal. I.e. the restriction to the $SL(2,\overline{\mathbb{Q}}_\ell)$ factor is trivial and the induced parameter:

$$\phi_\pi : W_L \to GSp_4(\overline{\mathbb{Q}}_\ell)$$

is irreducible.

Remarks

1. Given a supercuspidal parameter $\phi : W_L \to GSp_4(\overline{\mathbb{Q}}_\ell)$, the L-packet $\Pi_\phi := LLC_G^{-1}(\phi)$ has size 1 or 2, we say that ϕ is stable or endoscopic, respectively.
2. Let $std : GSp_4(\overline{\mathbb{Q}}_\ell) \to GL_4(\overline{\mathbb{Q}}_\ell)$ be the standard embedding.
   - (stable) $std \circ \phi$ is irreducible.
   - (endoscopic) $std \circ \phi \simeq \phi_1 \oplus \phi_2$, with ϕ_1 and ϕ_2 distinct irreducible 2-dimensional reps of GL_2 such that $det(\phi_1) = det(\phi_2)$.

The Main Theorem

Theorem (H)

- Assume $L/\mathbb{Q}_p$ is an unramified extension.
- $p > 2$.
- π is a representation of $G = GSp_4$ or $GU_2(D)$ such that the Gan-Takeda or Gan-Tantono parameter ϕ_π is supercuspidal, respectively.
- Then we have an equality: $\phi_\pi = \phi_\pi^{FS}$

Remark

The assumption that $L/\mathbb{Q}_p$ is unramified and that $p > 2$ is needed to apply basic uniformization for Shimura varieties of abelian type, as proven by Shen.

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Rappaport-Zink Spaces

A key insight of Harris and Taylor was to geometrically realize the Local Langlands correspondence for $G = GL_n$ in the cohomology of Rapoport-Zink spaces. The generic fiber of these spaces at infinite level can be reinterpreted as shtuka spaces, which in turn relate to the Fargues-Scholze LLC.

Definition

We set $k := \overline{\mathbb{F}}_p$, $\breve{\mathbb{Z}}_p := W(k)$. We consider $\mathbb{X}$ a p-divisible group over k of dimension d and height n. We consider the formal scheme:

$$\mathcal{M}_{\mathbb{X}}/Spf(\breve{\mathbb{Z}}_p)$$

parametrizing, for $S/Spf(\breve{\mathbb{Z}}_p)$, pairs (X, ρ), where X/S is a p-divisible group, and $\rho : X \times_S \overline{S} \to \mathbb{X} \times_k \overline{S}$ is a quasi-isogeny, where $\overline{S} = S \times_{Spf(\breve{\mathbb{Z}}_p)} Spec(k)$ is the special fiber.

Lubin-Tate Space

Example

In the case, that $d = 1$, we recover the Lubin-Tate space parametrizing $\mathbb{Z}$ disjoint copies of the space of deformations of a 1-dimensional formal group with height n:

$$LT_n := \mathcal{M}_{\mathbb{X}} \simeq \sqcup_{\mathbb{Z}} Spf(\breve{\mathbb{Z}}_p[|T_1, \ldots, T_{n-1}|])$$

Denote by $\mathcal{M}_{\mathbb{X},\breve{\mathbb{Q}}_p}$, the adic generic fiber of this moduli space, and let $\mathcal{M}_{\mathbb{X},\infty,\breve{\mathbb{Q}}_p}$ be the moduli space at infinite level.
Scholze-Weinstein provide a moduli interpretation of this space in terms of shtuka spaces on the Fargues-Fontaine curve.

Definition

1. Let $Perf$ be the category of perfectoid spaces in characteristic p.
2. For any $S \in Ob(Perf)$, we have an associated relative Fargues-Fontaine curve X_S.

Remark

X_S has the property that its primitive Cartier divisors correspond to characteristic 0 untilts of S.

Theorem (Fargues)

1. Fix an algebraically closed complete field F of characteristic p.
2. Then G-bundles on $X := X_F$ correspond to elements of the Kottwitz set $B(G) := G(\breve{\mathbb{Q}}_p)/(b \sim gb\sigma(g)^{-1})$, where σ is the Frobenius on $\breve{\mathbb{Q}}_p$. In other words, we have an isomorphism:

$$B(G) \xrightarrow{\simeq} |Bun_G|$$

$$b \mapsto |Bun_G^b| = *$$

Remark

Elements of $B(G)$ parametrize G-isocrystals over k

Definition

1. Let $\mu \in X_*(G_{\overline{\mathbb{Q}}_p})^+$ be a minuscule cocharacter with reflex field E.
2. Let $b \in B(G, \mu) \subset B(G)$ be the unique basic element. We call the triple (G, b, μ) a local Shimura datum.
3. $\breve{E} = E\breve{\mathbb{Q}}_p$
4. Let $\mathcal{E}_b$ be the bundle on X corresponding to $b \in B(G)$.
5. We define the diamond $Sh(G, b, \mu)_\infty / Spd(\breve{E})$, parametrizing for $S \in Ob(Perf)$, with a map $S \to Spd(\breve{E})$ and associated untilt $S^\sharp$ over $\breve{E}$, modifications:

$$\mathcal{E}_{0,S} \to \mathcal{E}_{b,S}$$

at the Cartier divisor $S^\sharp$ with meromorphy bounded by μ.

Remark

The space $Sh(G, b, \mu)_\infty$ has commuting actions by $G(\mathbb{Q}_p) = Aut(\mathcal{E}_0)$ and $J_b(\mathbb{Q}_p) = Aut(\mathcal{E}_b)$, where J_b is the σ-centralizer of $b \in G(\breve{\mathbb{Q}}_p)$. We have similar actions on $\mathcal{M}_{\mathbb{X},\infty,\breve{\mathbb{Q}}_p}$.

Theorem (Scholze-Weinstein)

For the fixed p-divisible group $\mathbb{X}/k$ of dimension d and height n as before, let $\mu = (1, \ldots, 1, 0, \ldots, 0)$ be the minisicule cocharacter of GL_n with d 1s and $b \in B(G)$ be the element corresponding to the isogeny class of $\mathbb{X}/k$. Then we have a $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$-equivariant isomorphism of diamonds over $Spd(\breve{\mathbb{Q}}_p)$:

$$\mathcal{M}^\diamond_{\mathbb{X},\infty,\breve{\mathbb{Q}}_p} \simeq Sh(G, \mu, b)_\infty$$

The Cohomology of the Lubin-Tate Tower

Example

Suppose that $\mathbb{X}$ has dimension 1 and height n, so that $\mu = (1, 0 \ldots, 0)$, then $LT^{\diamond}_{n,\infty} \simeq Sh(GL_n, \mu, b)_\infty$ parametrizes injections of the form:

$$\mathcal{O}_X^n \hookrightarrow \mathcal{O}(\frac{1}{n})$$

with cokernel of length 1 supported at a single closed point of X, where $\mathcal{O}(\frac{1}{n})$ is the rank n bundle on X of slope $\frac{1}{n}$ and $\mathcal{E}_0$ is the trivial bundle of rank n. In this case, $J_b(\mathbb{Q}_p) = D^*_{1/n}$, where $D_{1/n}$ is the division algebra of invariant $1/n$.

The Cohomology of the Lubin-Tate Tower

Definition

1. Let (G, b, μ) be a local Shimura datum.
2. Let $\pi \in \Pi(G)$ and $\rho \in \Pi(J_b)$.
3. Let $\mathcal{H}(G) := C_c^\infty(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ and $\mathcal{H}(J_b) := C_c^\infty(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ denote the usual smooth Hecke algebras.
4. Set $R\Gamma_c(G, b, \mu) := R\Gamma_c(Sh(G, b, \mu)_\infty, \overline{\mathbb{Q}}_\ell)$.
5. Set $R\Gamma_c(G, b, \mu)[\pi] := R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(G)} \pi$ and $R\Gamma_c(G, b, \mu)[\rho] := R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \rho$.

The Cohomology of the Lubin-Tate Tower

Theorem (Harris-Taylor)

1. Let $(GL_n, b, (1, 0, \ldots, 0))$ be the Local Shimura datum defining the Lubin-Tate tower.
2. Let π be a supercuspidal representation of $G(\mathbb{Q}_p)$.
3. Let $\rho := JL^{-1}(\pi) \in \Pi(D^*_{\frac{1}{n}})$.
4. Then $R\Gamma_c(G, b, \mu)[\pi]$ is concentrated in middle degree $n - 1$, and its cohomology decomposes as a $J_b(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$ representation as:

$$\rho \boxtimes \phi_\pi \otimes |\cdot|^{(1-n)/2}$$

where $\phi_\pi \in \Phi^W(G)$ is the semi-simplified Weil parameter associated to π by Harris-Taylor.

Remarks

1. The idea behind the proof of this Theorem is to relate this cohomology to the cohomology of a global Shimura variety associated to an inner form of $GL_n/\mathbb{Q}$, via basic uniformization, and appeal to Global Results.
2. Using this dictionary between Shtukas and p-divisible groups, we will see that this theorem gives rise to the compatibility with the Fargues-Scholze correspondence.
3. In a similar fashion, we will realize the Gan-Takeda/Gan-Tantono parameter in the cohomology of $R\Gamma_c(G, b, \mu)$, where $G = GSp_4$ and $\mu = (1, 1, 0, 0)$ is the Siegel cocharacter.

Function-Sheaf Dictionary

To a diamond or v-stack X, Fargues-Scholze define a triangulated category $D(X) := D_{lis}(X, \overline{\mathbb{Q}}_\ell)$

Proposition (Fargues-Scholze)

1. For a connected reductive group $G/\mathbb{Q}_p$, define the v-stack $B\underline{G(\mathbb{Q}_p)} := [*/\underline{G(\mathbb{Q}_p)}]$.
2. We have an equivalence of categories:

$$D(B(\underline{G(\mathbb{Q}_p)})) \xrightarrow{\simeq} D(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$$

where the RHS is the unbounded derived category of smooth $\overline{\mathbb{Q}}_\ell$-representations of $G(\mathbb{Q}_p)$ and Verdier duality corresponds to smooth duality.

Function-Sheaf Dictionary

- We have an open immersion:

$$j : B\underline{G(\mathbb{Q}_p)} \simeq Bun_G^{\mathbf{1}} \hookrightarrow Bun_G$$

  given by the inclusion of the HN-strata of Bun_G corresponding to the trivial bundle.
- Given $\pi \in \Pi(G)$, we can use the previous proposition to construct a sheaf:

$$j_!(\mathcal{F}_\pi) \in D(Bun_G)$$

- Following V.Lafforgue, Fargues-Scholze construct an excursion algebra acting on $D(Bun_G)$, which acts on this sheaf via eigenvalues determined by the parameter ϕ_π.

Hecke Operators

- For any finite set I, let X^I be the product of I-copies of the diamond $X = Spd(\breve{\mathbb{Q}}_p)/Frob^{\mathbb{Z}}$.
- We then have the Hecke-Stack:

$$\begin{tikzcd} & Hck \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow} \times supp"] & \\ Bun_G & & Bun_G \times X^I \end{tikzcd}$$

parametrzing for $S \in Ob(Perf)$ triples $(\mathcal{E}_0, \mathcal{E}_1, j, S_i^{\sharp} i \in I)$, where $j : \mathcal{E}_0 \to \mathcal{E}_1$ is a modification of two G-bundles on X_S away from the Cartier divisors defined by the untilts $S_i^{\sharp}$ for $i \in I$ of S.

Hecke Operators

- Given $W \in Rep_{\overline{\mathbb{Q}}_\ell}({}^L G^I)$, Geometric Satake furnishes a sheaf $\mathcal{S}_W$ on Hck.
- We then define the Hecke operator:

$$T_W : D(Bun_G) \to D(Bun_G \times X^I)$$

$$\mathcal{F} \mapsto R(h^{\rightarrow} \times supp)_*(h^{\leftarrow *}(\mathcal{F}) \otimes^{\mathbb{L}} \mathcal{S}_W)$$

Drinfeld's Lemma

Theorem (Fargues-Scholze)

The natural map:

$$Spd(\breve{\mathbb{Q}}_p)/Frob^{\mathbb{Z}} = X \to [*/W_{\mathbb{Q}_p}]$$

induces a pullback map:

$$D(Bun_G \times [*/\underline{W^I_{\mathbb{Q}_p}}]) \to D(Bun_G \times X^I)$$

is fully faithful and an equivalence if I is a singelton. The functor T_W takes values in:

$$D(Bun_G \times [*/\underline{W^I_{\mathbb{Q}_p}}])$$

Excursion Operators

Definition

1. Let $(I, W, (\gamma_i)_{i\in I}, \beta, \alpha)$ be the datum of:
   - A finite set I
   - A representation $W \in Rep_{\overline{\mathbb{Q}}_\ell}({}^L G^I)$
   - A tuple of elements $(\gamma_i)_{i\in I} \in W^I_{\mathbb{Q}_p}$
   - Maps of representations: $\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \Delta^* W \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell$
2. Given such a datum, one defines the excursion operator on $D(Bun_G)$ to be the endomorphism of the identity functor:

$$id = T_{\overline{\mathbb{Q}}_\ell} \xrightarrow{\alpha} T_{\Delta^* W} = T_W \xrightarrow{(\gamma_i)_{i\in I}} T_W = T_{\Delta^* W} \xrightarrow{\beta} T_{\overline{\mathbb{Q}}_\ell} = id$$

Excursion Operators

Via looking at the action of this natural transformation on the triangulated sub-category:

$$D(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \simeq D(B\underline{G(\mathbb{Q}_p)}) \subset D(Bun_G)$$

we can apply V. Lafforgue's reconstruction theorem to deduce:

Theorem (Fargues-Scholze)

Given $\pi \in \Pi(G)$, there is a unique continuous semisimple map:

$$\phi_\pi^{FS} : W_{\mathbb{Q}_p} \to^L G(\overline{\mathbb{Q}}_\ell)$$

characterized by the property that the action on π by an excursion operator is the composite:

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \Delta^* W = W \xrightarrow{(\phi_\pi^{FS}(\gamma_i))_{i \in I}} W = \Delta^* W \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell$$

Upshot

- Let's fix $\pi \in \Pi(G)$ and let (G, b, μ) be a local Shimura datum with E the reflex field of μ.
- Consider the case where $I = \{*\}$ and $W = V_\mu$ is a highest weight rep of highest weight μ. The sheaf $\mathcal{S}_W$ is then supported on Hck_μ the subspace parametrizing modifications of type μ, $\mathcal{S}_W \simeq \overline{\mathbb{Q}}_\ell[d](\frac{d}{2})$, where $d = \langle 2\rho_G, \mu\rangle$.
- We look at the Hecke-Correspondence:

$$\begin{tikzcd}
 & & Hck_\mu \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow} \times supp"] & & \\
Bun_G^{\mathbf{1}} \arrow[r, "j"] & Bun_G & & Bun_G \times X_E & Bun_G^b \times X_E \arrow[l, "j_b \times id"']
\end{tikzcd}$$

- We consider the sheaf:

$$(j_b \times id)^* T_W(j_!(\mathcal{F}_\pi)) \in D(J_b(\mathbb{Q}_p) \times W_E, \overline{\mathbb{Q}}_\ell)$$

Upshot

- The simultaneous fibers of Hck_{μ} over $Bun_G^{\mathbf{1}}$ and Bun_G^b is given by:
$$[Gr^b_{G,\mu}/\underline{G(\mathbb{Q}_p)}]$$
where $Gr^b_{G,\mu} \subset Gr_{G,\mu}$ is the open Newton strata in the Schubert cell of the B_{dR}^+ -affine Grassmanian parametrizing modifications $\mathcal{E}_0 \to \mathcal{E}$ of type μ, where $\mathcal{E} \simeq \mathcal{E}_b$, after pulling back to each geometric point.
- The Shtuka space sits as $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$-torsor over this space:
$$Sh(G,b,\mu)_{\infty} \to [Gr^b_{G,\mu}/\underline{G(\mathbb{Q}_p)}]$$
This gives rise to the key identification:
$$R\Gamma_c(G,b,\mu)[\pi][d](\frac{d}{2}) \simeq (j_b \times id)^* T_W(j_!(\mathcal{F}_{\pi}))$$

Compatibility for $G = GL_n$

- Let $(G, b, \mu) = (GL_n, b, (1, 0, \ldots, 0))$ be the Shimura datum defining the Lubin-Tate tower. Then $d = n - 1$ and the work of Harris-Taylor combined with the previous result gives us an isomorphism:

$$(j_b \times id)^* T_W(j_!(\mathcal{F}_\pi)) \simeq R\Gamma_c(G, b, \mu)[\pi][d](\frac{d}{2}) \simeq \rho \boxtimes \phi_\pi$$

  where $\rho := JL^{-1}(\pi)$.
- However, it follows from the Fargues-Scholze construction that the LHS should be valued in the category generated by ϕ_π^{FS}, so in particular $\phi_\pi^{FS} = \phi_\pi$.

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Compatibility for $G = GSp_4$

- Let $L/\mathbb{Q}_p$ be a finite extension and $(G, b, \mu) := (Res_{L/\mathbb{Q}_p}(GSp_4), b, (1, 1, 0, 0))$, where $b \in B(G, \mu)$ is the unique basic element.
- $J_b = Res_{L/\mathbb{Q}_p} GU_2(D)$, where D is the quaternionic division algebra over L.
- Note that the rep $\hat{G} \simeq GSp_4$ induced by μ is the representation:

$$std : GSp_4 \to GL_4$$

  and that $\langle 2\rho_G, \mu \rangle = 3$.
- Given $\pi \in \Pi(GSp_4)$, with supercuspidal Gan-Takeda parameter ϕ_π, compatibility should follow from showing that:

$$R\Gamma_c(G, b, \mu)[\pi]$$

  is concentrated in degree 3 and is $std \circ \phi_\pi \otimes |\cdot|^{-\frac{3}{2}}$-isotypic.

Comaptibility for $G = GSp_4$

- One issue is that $std \circ \phi_\pi$ is not enough to uniquely determine the parameter ϕ_π. However, if one also includes the datum of a W_L-invariant symplectic similitude pairing on $std \circ \phi_\pi$ it is.
- Moreover, note that, since the excursion algebra is acting on $R\Gamma_c(G, b, \mu)[\pi]$ as a complex of $J_b(\mathbb{Q}_p)$-representations via scalars, it suffices to identify $std \circ \phi_\pi$ with its symplectic similitude pairing on a non-zero sub-quotient.

This leads us to the key Proposition:

- Assume from now on that $p > 2$ and $L/\mathbb{Q}_p$ is unramified.

Proposition (H)

- Assume that $\rho \in \Pi(GU_2(D))$ has supercuspidal Gan-Tantono parameter ϕ_ρ.
- Let $R\Gamma_c(G, b, \mu)[\rho]_{sc}$ denote the sub-complex where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation.
- $R\Gamma_c(G, b, \mu)[\rho]_{sc}$ is non-zero concentrated in degree 3 and is $std \circ \phi_\rho \otimes |\cdot|^{-\frac{3}{2}}$-isotypic as a W_L-representation.
- $R\Gamma_c(G, b, \mu)[\rho]_{sc}$ carries a non-degenerate W_L-invariant symplectic similitude form.

Applications

- The previous proposition, through the formal considerations sketched before, gives compatibility for $\rho \in \Pi(GU_2(D))$ with supercuspidal Gan-Tantono parameter.
- We would like to apply this compatibility together with the previous description of the W_L-action to give a full description of $R\Gamma_c(G, b, \mu)[\rho]$.
- We set:

$$R\Gamma_c^{KW}(G, b, \mu)[\rho] := R\mathcal{H}om_{J_b(\mathbb{Q}_p)}(R\Gamma_c(G, b, \mu), \rho)[-2d](-d)$$

Kaletha-Weinstein give a general description of this in the Grothendieck Group of admissible representations.

Applications

Theorem (Kaletha-Weinstein)

For $\rho \in \Pi(GU_2(D))$ with supercuspidal Gan-Tantono parameter.

- Let $S_{\phi_\rho} = Cent(\phi_\rho, \hat{G})$.
- Let $\Pi_{\phi_\rho}(GSp_4)$ denote the L-packet over ϕ_ρ.
- Then the cohomology of $R\Gamma_c^{KW}(G, b, \mu)[\rho]$ is valued in admissible $G(\mathbb{Q}_p)$-representations and we have the following equality in $K_0(G(\mathbb{Q}_p))$ the Grothendieck group of admissible $G(\mathbb{Q}_p)$-modules:

$$[R\Gamma_c^{KW}(G, b, \mu)[\rho])] = -\sum_{\pi \in \Pi_{\phi_\rho}(G)} Hom_{S_{\phi_\rho}}(\delta_{\pi,\rho}, std \circ \phi_\rho)\pi$$

where $\delta_{\pi,\rho}$ is the algebraic representation of S_{ϕ_ρ} defined by the refined Local Langlands correspondence for $GU_2(D)$.

Applications

The Refined LLC

- Note that we have $Z(\hat{G}) = GL_1$.
- The refined Local Langlands correspondence defines a bijection:

$$\Pi_\phi(GSp_4) \leftrightarrow \{\text{irred. reps } \tau \text{ of } S_\phi \text{ s.t } \tau|_{Z(\hat{G})} = \mathbf{1}\}$$

- And a bijection:

$$\Pi_\phi(GU_2(D)) \leftrightarrow \{\text{irred. reps } \tau \text{ of } S_\phi \text{ s.t } \tau|_{Z(\hat{G})} = id_{GL_1}\}$$

- These bijections are characterized by the character identities proven by Gan-Chan, after fixing $(B, \psi) =: \mathfrak{m}$ for GSp_4.

Applications

The Refined LLC

- If ϕ is a stable supercuspidal parameter (i.e $std \circ \phi$ is irreducible) then the L-packet are singletons and $S_\phi = Z(\hat{G}) = GL_1$.
- If ϕ is an endoscopic parameter (i.e $std \circ \phi \simeq \phi_1 \oplus \phi_2$, with ϕ_i distinct 2-dimensional irreducibles and $det(\phi_1) = det(\phi_2)$) This gives rise to an identification:

  $$S_\phi = \{(a, b) \in GL_1 \times GL_1 | a^2 = b^2\} \subset GL_2 \times GL_2 \subset GL_4$$

  where the $Z(\hat{G}) = GL_1$ embeds diagonally.
- We see that $\pi_0(S_\phi) \simeq \mathbb{Z}/2\mathbb{Z}$. The L-packet of $\Pi_\phi(GSp_4) = \{\pi^+, \pi^-\}$ is indexed by the reps τ_{π^+} and τ_{π^-} of S_ϕ defined by the trivial and non-trivial character of $\pi_0(S_\phi)$.

Applications

- The L-packet $\Pi_\phi(GU_2(D)) = \{\rho_1, \rho_2\}$ is indexed by the representations τ_{ρ_1} and τ_{ρ_2} corresponding to projection to the two GL_1s.

Definition

Given $\pi \in \Pi_\phi(GSp_4)$ and $\rho \in \Pi_\phi(GU_2(D))$, we set:

$$\delta_{\pi,\rho} := \tau_\pi^\vee \otimes \tau_\rho$$

where $\tau_\pi^\vee$ denotes the contragradient.

Applications

- With these identifications pinned down, we can now write the down the RHS of the Kottwitz conjecture:

$$- \sum_{\pi \in \Pi_{\phi_\rho}(G)} Hom_{S_{\phi_\rho}}(\delta_{\pi,\rho}, std \circ \phi_\rho)\pi$$

with $\delta_{\pi,\rho}$ as above.

Corollary

If $\rho \in \Pi(GU_2(D))$ has a stable supercuspidal Gan-Tantono parameter ϕ_ρ, we have that:

$$[R\Gamma_c^{KW}(G, \mu, b)[\rho]] = -Hom_{id}(id, std \circ \phi_\rho)\pi = -4\pi$$

where $\Pi_{\phi_\rho}(GSp_4) = \{\pi\}$.

Applications

Corollary

If $\rho = \rho_i \in \Pi(GU_2(D))$ has endoscopic Gan-Tantono parameter, we have that $[R\Gamma_c^{KW}(G, \mu, b)[\rho]]$ is equal to:

$$-Hom_{S_{\phi_\rho}}(\tau_{\rho_i}, std \circ \phi_\rho)\pi^+ + -Hom_{S_{\phi_\rho}}(\tau_{\rho_{3-i}}, std \circ \phi_\rho)\pi^-$$

Writing $std \circ \phi \simeq \phi_1 \oplus \phi_2$ this identifies with:

$$-Hom_{id}(id, \phi_i)\pi^+ + -Hom_{id}(id, \phi_{3-i})\pi^- = -2\pi^+ + -2\pi^-$$

where $\Pi_{\phi_\rho} = \{\pi^+, \pi^-\}$.

Applications

Theorem (H.)

- For any $\rho \in \Pi(GU_2(D))$ such that the Gan-Tantono parameter ϕ_ρ is supercuspidal, $R\Gamma_c(G, b, \mu)[\rho]$ is concentrated in degree 3 and is isomorphic to $R\Gamma_c^{KW}(G, \mu, b)[\rho]$.
- If ρ is stable supercuspidal, we have an isomorphism of $G(\mathbb{Q}_p) \times W_L$ representations:

$$H^3(R\Gamma_c(G, b, \mu)[\rho]) \simeq \pi \boxtimes std \circ \phi_\rho \otimes |\cdot|^{-\frac{3}{2}}$$

- If $\rho = \rho_i$ is endoscopic, we have an isomorphism of $G(\mathbb{Q}_p) \times W_L$ representations:

$$H^3(R\Gamma_c(G, b, \mu)[\rho]) \simeq \pi^+ \boxtimes \phi_i \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_{3-i} \otimes |\cdot|^{-3/2}$$

Applications

Proof.

We know from our previous result that the supercuspidal Gan-Tantono parameter ϕ_ρ is the same as the Fargues-Scholze parameter. The geometry of the Fargues-Scholze construction gives us information about the sheaves associated to the representations with such parameters. □

Definition

For G any connected reductive group, we let $j : Bun_G^{\mathbf{1}} \hookrightarrow Bun_G$ be the open immersion as before. We say a representation $\rho \in \Pi(G)$ is inert if the sheaf $\mathcal{F}_\rho$ satisfies that the natural map:

$$j_!(\mathcal{F}_\rho) \xrightarrow{\simeq} Rj_*(\mathcal{F}_\rho)$$

is an isomorphism.

Applications

Theorem (Fargues-Scholze)

If ρ has supercuspidal Fargues-Scholze parameter then ρ is inert.

Proof.

If $(Res_{L/\mathbb{Q}_p}(GSp_4), b, \mu)$ is the local Shimura datum from before, the dual Shimura datum is given by $(Res_{L/\mathbb{Q}_p}(GU_2(D)), \hat{b}, \mu^{-1})$. Applying our key isomorphism, we get:

$$R\Gamma_c(G, b, \mu)[\rho][d](\frac{d}{2}) \simeq$$

$$R\Gamma_c(Res_{L/\mathbb{Q}_p}(GU_2(D)), \mu^{-1}, \hat{b})[\rho][d](\frac{d}{2}) \simeq (j_{\hat{b}} \times id)^* T_{\mu^{-1}} j_!(\mathcal{F}_\rho)$$

$\square$

Applications

Proof.

We now apply Verdier duality to both sides of:

$$R\Gamma_c(G,b,\mu)[\rho][d](\frac{d}{2}) \simeq (j_{\hat{b}} \times id)^* T_{\mu^{-1}} j_!(\mathcal{F}_\rho)$$

The RHS becomes isomorphic to:

$$\mathbb{D}(j_{\hat{b}} \times id)^* T_{\mu^{-1}} j_!(\mathcal{F}_\rho) \simeq (j_{\hat{b}} \times id)^* T_{\mu^{-1}} j_!(\mathbb{D}(\mathcal{F}_\rho)) \simeq (j_{\hat{b}} \times id)^* T_{\mu^{-1}} j_!(\mathcal{F}_{\rho^*}))$$

$$\simeq R\Gamma_c(G,b,\mu)[\rho^*][d](\frac{d}{2})$$

while the LHS is acted on by Verdier duality on $Sh(G,\mu,b)_\infty$, which is a cohomologically smooth diamond of dimension d and so the dualizing object is isomorphic to $\overline{\mathbb{Q}}_\ell[2d](d)$ □

Applications

Proof.

- This allows us to deduce that:

$$R\mathcal{H}om(R\Gamma_c(G,b,\mu)[\rho], \overline{\mathbb{Q}}_\ell) \simeq R\Gamma_c(G,b,\mu)[\rho^*][2d](d)$$

- In particular, we have a natural W_L-equivariant isomorphism of admissible $G(\mathbb{Q}_p)$-representations for all $0 \leq i \leq 2d$:

$$H^i(R\Gamma_c(G,b,\mu)[\rho])^* \simeq H^{2d-i}(R\Gamma_c(G,b,\mu)[\rho^*])(d)$$

- Moreover, the local Shimura datum occurs in the basic uniformization of a global Shimura variety. Applying global vanishing results for $(\mathfrak{g}, K)$-cohomology with regular weights and the previous duality theorem, one can show that this must be concentrated in middle degree 3.

Applications

Proof.

- We note that:

$$R\Gamma_c(G, b, \mu)[\rho^*][2d](d) \simeq$$

$$R\mathcal{H}om(R\Gamma_c(G, b, \mu)[\rho], \overline{\mathbb{Q}}_\ell) = R\mathcal{H}om(R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \rho, \overline{\mathbb{Q}}_\ell)$$

$$\simeq R\mathcal{H}om_{J_b(\mathbb{Q}_p)}(R\Gamma_c(G, b, \mu), \rho^*) = R\Gamma_c^{KW}(G, \mu, b)[\rho^*][2d](d)$$

- In particular, after taking contragradients and moving the shifts/Tate-Twists, we get:

$$R\Gamma_c(G, b, \mu)[\rho] \simeq R\Gamma_c^{KW}(G, b, \mu)[\rho]$$

$\square$

Applications

Proof.

- Assume ϕ_ρ is stable supercuspidal, for simplicity.
- We have, by Kaletha-Weinstein and the previous results, that:

$$[H^3(R\Gamma_c(G, b, \mu)[\rho])] = -4\pi$$

where $\{\pi\} = \Pi_{\phi_\rho}(GSp_4)$.
- If ω denotes the central character of ρ, $R\Gamma_c(G, b, \mu)[\rho]$ is a complex of admissible $G(\mathbb{Q}_p)$ representations with central character ω.

$\square$

Applications

Proof.

- π is supercuspidal, so it is injective/projective in the category of admissible representations with fixed central character. Therefore, as $G(\mathbb{Q}_p)$-representations:

$$H^3(R\Gamma_c(G,b,\mu)[\rho]_{sc}) = H^3(R\Gamma_c(G,b,\mu)[\rho]) \simeq \pi^{\oplus 4}$$

- However, our previous proposition tells us that $H^3(R\Gamma_c(G,b,\mu)[\rho]_{sc})$ is $std \circ \phi_\rho \otimes |\cdot|^{-\frac{3}{2}}$-isotypic.

$\square$

Applications

- What about $G = GSp_4$? Basic uniformization is not known to hold in this case because $GU_2(D)$ is non-split.

Theorem (H.)

For $\pi \in \Pi(GSp_4)$ with supercuspidal Gan-Takeda parameter ϕ_π, we have that:

$$\phi_\pi = \phi_\pi^{FS}$$

Applications

Proof.

- Our key isomorphism tells us that:

$$R\Gamma_c(G,b,\mu)[\pi][d](\frac{d}{2}) \simeq (j_b \times id)^* T_\mu j_!(\mathcal{F}_\pi)$$

- The excursion algebra commutes with Hecke operators/restriction to HN-strata. Therefore, it acts on the LHS via the parameter ϕ_π^{FS}.
- However, we have:

$$R\mathcal{H}om(R\Gamma_c(G,b,\mu)[\pi],\overline{\mathbb{Q}}_\ell) \simeq R\Gamma_c^{KW}(G,b,\mu)[\pi^*][2d](d)$$

$\square$

Applications

Proof.

- Applying Kaletha-Weinstein, will tell us that the excursion algebra acts on the LHS of our key isomorphism as:

$$\phi_{\rho}^{FS} = \phi_{\rho} = \phi_{\pi}$$

where the equalities follow by the character identities and our previous theorem. Therefore, $\phi_{\pi} = \phi_{\pi}^{FS}$.

$\square$

Applications

By applying formal properties from the Fargues-Scholze construction, together with our previous results we can deduce:

Theorem (H.)

- Assume $\pi \in \Pi(G)$ has supercuspidal Gan-Takeda parameter ϕ_π.
- Then $R\Gamma_c(G, b, \mu)[\pi]$ is concentrated in middle degree 3.
- If ϕ_π is stable supercuspidal, then the middle cohomology is isomorphic to:

$$\rho \boxtimes std \circ \phi_\pi \otimes |\cdot|^{-\frac{3}{2}}$$

as a $G(\mathbb{Q}_p) \times W_L$ representation, where $\{\rho\} = \Pi_{\phi_\pi}(GU_2(D))$.

Applications

Theorem (H.)

- If ϕ_π is endoscopic and $\pi = \pi^+$ then the middle cohomology is isomorphic to:

$$\rho_1 \boxtimes \phi_1 \otimes |\cdot|^{-\frac{3}{2}} \oplus \rho_2 \boxtimes \phi_2 \otimes |\cdot|^{-\frac{3}{2}}$$

  and if $\pi = \pi^-$ then it is isomorphic to:

$$\rho_1 \boxtimes \phi_2 \otimes |\cdot|^{-\frac{3}{2}} \oplus \rho_2 \boxtimes \phi_1 \otimes |\cdot|^{-\frac{3}{2}}$$

  where $std \circ \phi_\pi \simeq \phi_1 \oplus \phi_2$.

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1 The Main Theorem

2 The Fargues-Scholze LLC and compatibility for $G = GL_n$

3 Compatibility for $G = GSp_4$ and $GU_2(D)$ and Applications

4 Proof of the Key Proposition

We will now discuss the proof of the key Proposition:

Proposition (H)

- Assume that $\rho \in \Pi(GU_2(D))$ has supercuspidal Gan-Tantono parameter ϕ_ρ.
- Let $R\Gamma_c(G, b, \mu)[\rho]_{sc}$ denote the sub-complex where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation.
- $R\Gamma_c(G, b, \mu)[\rho]_{sc}$ is non-zero concentrated in degree 3 and is $std \circ \phi_\rho \otimes |\cdot|^{-\frac{3}{2}}$-isotypic as a W_L-representation.
- $R\Gamma_c(G, b, \mu)[\rho]_{sc}$ carries a non-degenerate W_L-invariant symplectic similitude form.

Basic Uniformization

The key idea will be to use basic uniformization to apply global results:

- Let $\mathbf{G}/\mathbb{Q}$ be a connected reductive group over the rational numbers.
- We let $(\mathbf{G}, X)$ be a Shimura datum of abelian type.
- We set $G = \mathbf{G}_{\mathbb{Q}_p}$ and, using the isomorphism $j : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$, we regard $X_{\mathbb{C}}^{-1}$ as a conjugacy class of cocharacters

  $$\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$$

  and consider the Kottwitz set $B(G, \mu)$.
- We let $b \in B(G, \mu)$ be the unique basic element and let J_b be the σ-centralizer of b.

Basic Uniformization

- For any compact open $K \subset \mathbf{G}(\mathbb{A}^f)$, we let $\mathcal{S}(\mathbf{G}, X)_K$ be the rigid analytic global Shimura variety over $\mathbb{C}_p$ of level K.
- We write $K = K^p K_p$ for compact opens $K_p \subset G(\mathbb{Q}_p)$ and $K^p \subset \mathbf{G}(\mathbb{A}^{f,p})$.
- We define:

$$\mathcal{S}(\mathbf{G}, X)_{K^p} = \lim_{K_p \to \{1\}} \mathcal{S}(\mathbf{G}, X)_{K^p K_p}$$

This is representable by a perfectoid space after completing the structure sheaf.

Basic Uniformization

- By results of Shen, there exists a canonical $G(\mathbb{Q}_p)$-equivariant Hodge-Tate period map:

$$\pi_{HT} : \mathcal{S}(\mathbf{G}, X)_{K^p} \to \mathcal{F}\ell_{G,\mu}$$

  where $\mathcal{F}\ell_{G,\mu} := (G_{\mathbb{C}_p}/P_\mu)^{ad}$.
- For $b \in B(G, \mu)$, we define the Newton strata $\mathcal{S}(\mathbf{G}, X)^b_{K^p}$ by pulling back the Newton strata $\mathcal{F}\ell^b_{G,\mu}$ along π_{HT}.

Basic Uniformization

Definition

We say a global Shimura datum $(\mathbf{G}, X)$ satisfies basic uniformization at p if there exists a unique up to isomorphism $\mathbb{Q}$-inner form $\mathbf{G}'$ of $\mathbf{G}$ satisfying:

1. $\mathbf{G}'_{\mathbb{A}^{\{p\infty\}}} \simeq \mathbf{G}_{\mathbb{A}^{\{p\infty\}}}$ as algebraic groups over $\mathbb{A}^{\{p\infty\}}$.
2. $\mathbf{G}'_{\mathbb{Q}_p} \simeq J_b$
3. $\mathbf{G}'(\mathbb{R})$ is compact modulo center.

such that...

Basic Uniformization

Definition

There is a $\mathbf{G}(\mathbb{A}^f)$-equivariant isomorphism of diamonds over $\mathbb{C}_p$:

$$\lim_{K^p} \mathcal{S}(\mathbf{G}, X)^b_{K^p} \simeq \underline{\mathbf{G}'(\mathbb{Q})} \backslash \underline{\mathbf{G}'(\mathbb{A}^f)} \times_{Spd(\mathbb{C}_p)} Sh(G, \mu, b)_\infty / \underline{J_b(\mathbb{Q}_p)}$$

such that

$$\pi_{HT} : \lim_{K^p} \mathcal{S}(\mathbf{G}, X)^b_{K^p} \to \mathcal{F}\ell^b_{G,\mu} \simeq Sh(G, \mu, b)_\infty / \underline{J_b(\mathbb{Q}_p)}$$

agrees with projection to the second factor. Moreover, we assume this descends to an isomorphism of diamonds over $\breve{E}$, where E is the reflex field of μ.

Basic Uniformization

Theorem (Shen)

If $(\mathbf{G}, X)$ is a Shimura datum of abelian type and $p > 2$ is a prime where $G_{\mathbb{Q}_p}$ is unramified then $(\mathbf{G}, X)$ satisifies basic uniformization at p.

Basic Uniformization

From now on, we let:

- $F/\mathbb{Q}$ be a totally real field such that:
  - p is totally inert in F and $F_p \simeq L$.
- $\mathbf{G}$ a $\mathbb{Q}$-inner form of $Res_{F/\mathbb{Q}} GSp_4 =: \mathbf{G}^*$ such that:
  - $\mathbf{G}_{\mathbb{Q}_p} \simeq Res_{L/\mathbb{Q}_p} GSp_4 = G$
- Let ξ be the highest weight of an algebraic representation of $\mathbf{G}$.
- The isomorphism $i : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ determines a $\overline{\mathbb{Q}}_\ell$ local system on $\mathcal{S}(\mathbf{G}, X)_{K^p}$, denoted $\mathcal{L}_\xi$.
- We now consider the space of Gross's algebraic automorphic forms valued in this algebraic representation:

$$\mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}^f)/K^p, \mathcal{L}_\xi)$$

Basic Uniformization

Under the assumptions on L and p, we can apply the theorem of Shen to $(\mathbf{G}, X)$ to deduce:

Corollary

There exists a $G(\mathbb{Q}_p) \times W_L$-invariant map:

$$\Theta : R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash \mathbf{G}'(\mathbb{A}^f)/K^p, \mathcal{L}_\xi) \to$$

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)$$

where $(G, b, \mu) = (Res_{L/\mathbb{Q}_p} GSp_4, b, (1,1,0,0))$ is the local Shimura datum from before.

Basic Uniformization

Proposition

Let $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$ denote the part of the cohomology where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation. Then Θ induces an isomorphism:

$$\Theta : R\Gamma_c(G, b, \mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}^f)/K^p, \mathcal{L}_{\xi,E}) \simeq$$

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$$

of $G(\mathbb{Q}_p) \times W_L$-representations.

Global Input

We assume, for simplicity, that $[F:\mathbb{Q}]$ is odd. We fix a totally inert prime q in F and assume that $\mathbf{G}$ satisifies:

1. $\mathbf{G}(\mathbb{R}) \simeq GSp_4(\mathbb{R}) \times GU_2(\mathbb{H})^{[F:\mathbb{Q}]-1}$
2. $\mathbf{G}_{F_v} \simeq GSp_4/F_v$ at all finite places v.

Global Input

- We let ρ be a representation of $J_b(\mathbb{Q}_p) = GU_2(D)(L)$, with supercuspidal Gan-Tantono parameter.
- Let $\mathbf{G}'$ be the inner form of $\mathbf{G}$ furnished by basic uniformization.
- Using the simple trace formula, for sufficiently large ξ, we can choose a globalization of a character twist of ρ to a cuspidal automorphic representation of π' of $\mathbf{G}'$ which occurs as a $J_b(\mathbb{Q}_p)$-stable direct summand of:

$$\mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}^f)/K^p, \mathcal{L}_\xi)$$

  the space of algebraic automorphic forms, satisfying the condition that π'_q is an unramified twist of the Steinberg representation, where K^p is hyperspecial away from pq. Let $S = \{p, q\}$.

Global Input

- The previous isomorphism then furnishes a $G(\mathbb{Q}_p) \times W_L$-invariant split injection:

$$\Theta_\rho : R\Gamma_c(G, b, \mu)[\rho]_{sc} \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$$

We now prove the following theorem:

Theorem (H.)

There exists a globally generic cuspidal automorphic representation τ of $\mathbf{G}^* = Res_{F/\mathbb{Q}} GSp_4$ such that:

- At q, τ_q is an unramified twist of the Steinberg representation.
- At p, τ_p has Gan-Takeda parameter ϕ_ρ.
- At ∞, τ_∞ is cohomological of weight ξ.
- $\pi'^S \simeq \tau^S$.

Global Input

We can then apply the following theorem of Sorensen to the transfer τ:

Theorem

There exists, a unique (after fixing an isomorphism $i : \mathbb{C} \xrightarrow{\simeq} \overline{\mathbb{Q}}_\ell$) irreducible continuous representation $\rho_\tau : Gal(\overline{F}/F) \to GSp_4(\overline{\mathbb{Q}}_\ell)$ characterized by the property that, for each finite place $v \nmid \ell$ of F, we have:

$$WD(\rho_{\tau,i}|_{W_{F_v}})^{F-s.s} \simeq \phi_{\tau_v} \otimes |\cdot|^{-3/2}$$

where ϕ_{τ_v} is the semi-simplified Gan-Takeda parameter associated to τ_v.

Global Input

- Under the split injection:

$$\Theta_\rho : R\Gamma_c(G, b, \mu)[\rho]_{sc} \to R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$$

  will map to the $\pi'^{p\infty}$-isotypic part, where $\pi'^{p\infty}$ is regarded as a representation of $\mathbf{G}'(\mathbb{A}^{p\infty}) \simeq \mathbf{G}(\mathbb{A}^{p\infty})$.
- Kret-Shin show that the $\pi'^{p\infty}$-isotypic part is concentrated in degree 3 and compute the traces of Frobenius as a Galois representation in terms of the Langlands parameters of τ.
- This allows us to conclude that $R\Gamma_c(G, b, \mu)[\rho]_{sc}$ concentrated in degree 3 and is $std \circ \rho_\tau|_{W_{\mathbb{Q}_p}} = std \circ \phi_\rho \otimes |\cdot|^{-\frac{3}{2}}$-isotypic.

Global Input

- It remains to see that $R\Gamma_c(G,b,\mu)_{sc}$ is concentrated in degree non-zero and carries a non-degenerate W_L-invariant symplectic similitude form.
- Using the identification:

$$R\mathcal{H}om(R\Gamma_c(G,b,\mu)[\rho]_{sc},\overline{\mathbb{Q}}_\ell) \simeq R\Gamma_c^{KW}(G,b,\mu)[\rho^*]_{sc}[2d](d)$$

We can apply the work of Kaletha-Weinstein to conclude this is non-zero.

The Zelevinsky Involution

- We consider the Zelevinsky involution:

$$Zel : D(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \to D(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$$

$$A \mapsto R\mathcal{H}om_{G(\mathbb{Q}_p)}(A, \mathcal{H}(G))$$

- Fargues-Scholze extend this to an involution:

$$\tilde{Zel} : D(Bun_G) \to D(Bun_G)$$

which induces the Zelevinsky involution on the basic HN-strata under the function-sheaf dictionary and commutes with Hecke operators.

The Zelevinsky Involution

With this, and our key isomorphism, one can deduce the existence of an isomorphism:

$$Zel(R\Gamma_c(G,b,\mu)[\rho][d](\frac{d}{2})) \simeq R\Gamma_c(G,b,\mu)[Zel(\rho)][d](\frac{d}{2})$$

which induces an isomorphism:

$$Zel(R\Gamma_c(G,b,\mu)[\rho]_{sc}[d](\frac{d}{2})) \simeq R\Gamma_c(G,b,\mu)[Zel(\rho)]_{sc}[d](\frac{d}{2})$$

Then we have an identification:

$$R\Gamma_c(G,b,\mu)[Zel(\rho)]_{sc}[d](\frac{d}{2}) \simeq R\Gamma_c(G,b,\mu)[\rho^*]_{sc}[d](\frac{d}{2})$$

However, we have $\rho^* \simeq \rho \otimes \chi$, where χ is a character. So we get a non-degenerate W_L-invariant pairing:

$$R\Gamma_c(G,b,\mu)[\rho]_{sc}[d](\frac{d}{2}) \otimes^{\mathbb{L}}_{\mathcal{H}(G)} R\Gamma_c(G,b,\mu)[\rho\otimes\chi]_{sc}[d](\frac{d}{2}) \to \overline{\mathbb{Q}}_\ell$$

The Zelevinsky Involution

Using work of Chen on the connected components of $Sh(G, \mu, b)_\infty$, we can see that:

$$R\Gamma_c(G, b, \mu)[\rho \otimes \chi]_{sc} \simeq R\Gamma_c(G, b, \mu)[\rho]_{sc} \otimes \chi$$

Thus, we get a non-degenerate W_L-invariant pairing of complexes:

$$R\Gamma_c(G, b, \mu)[\rho]_{sc} \otimes^{\mathbb{L}} R\Gamma_c(G, b, \mu)[\rho]_{sc} \to \chi^{-1}[-2d](-d)$$

which induces the desired non-degenerate W_L-invariant symplectic similitude pairing in degree $3 = d$.

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