# Unipotent Categorical Local Langlands Correspondence

Tamir Hemo
(Joint with X. Zhu)

Department of Mathematics
California Institute of Technology

May 27, 2021

# Outline

# Introduction

Notations:

- $F$ a non-archimedean local field, e.g. $\mathbb{Q}_p$, $\mathbb{F}_q((t))$
- $G/F$ connected reductive group
- $\ell \neq p$ a prime

**Local Langlands correspondence**: Classify irreducible smooth representations of $G(F)$ in terms of Langlands parameters

$$\{\varphi\colon W_F \to {}^L G(\overline{\mathbb{Q}_\ell})\}/\hat{G}$$

# Introduction

**Categorical local Langlands correspondence**: Describe the derived category $\mathrm{Rep}(G(F), \overline{\mathbb{Q}_\ell})$ of smooth representations in terms of Galois data.

Hope: an object $V \in \mathrm{Rep}(G(F), \overline{\mathbb{Q}_\ell})$ should correspond to a quasi-coherent sheaf over a stack classifying families of Langlands parameters.

# Introduction

**Categorical local Langlands correspondence**: Describe the derived category $\mathrm{Rep}(G(F), \overline{\mathbb{Q}_\ell})$ of smooth representations in terms of Galois data.

Hope: an object $V \in \mathrm{Rep}(G(F), \overline{\mathbb{Q}_\ell})$ should correspond to a quasi-coherent sheaf over a stack classifying families of Langlands parameters.

Related work:

1. Emerton-Helm: local correspondence for $\mathrm{GL}_n$ in families.
2. Fargues-Scholze: bundles on the Fargues-Fontaine curve
3. Hellmann
4. Ben-Zvi, Chen, Helm, Nadler: Coherent Springer theory

# Introduction

The category $\mathrm{Rep}(G(F), \overline{\mathbb{Q}_\ell})$ needs to be modified

# Introduction

The category $\mathrm{Rep}(G(F), \overline{\mathbb{Q}_\ell})$ needs to be modified

This can be seen from several points of view:

# Introduction

The category $\mathrm{Rep}(G(F), \overline{\mathbb{Q}_\ell})$ needs to be modified

This can be seen from several points of view:

**Local Langlands correspondence (Vogan, Bernstein, Kaletha):**
Better to study representation theory of $G(F)$ together with that of its extended pure inner forms.

# Introduction

The category $\mathrm{Rep}(G(F), \overline{\mathbb{Q}_\ell})$ needs to be modified

This can be seen from several points of view:

**Local Langlands correspondence (Vogan, Bernstein, Kaletha):** Better to study representation theory of $G(F)$ together with that of its extended pure inner forms.

**Geometric Langlands (Drinfeld, Gaitsgory):** the "local Langlands category" should be the categorical Frobenius fixed points of the category of $\overline{\mathbb{Q}_\ell}$-sheaves on the loop group of $G$.

# Introduction

The category $\mathrm{Rep}(G(F), \overline{\mathbb{Q}_\ell})$ needs to be modified

This can be seen from several points of view:

**Local Langlands correspondence (Vogan, Bernstein, Kaletha):** Better to study representation theory of $G(F)$ together with that of its extended pure inner forms.

**Geometric Langlands (Drinfeld, Gaitsgory):** the "local Langlands category" should be the categorical Frobenius fixed points of the category of $\overline{\mathbb{Q}_\ell}$-sheaves on the loop group of $G$.

**Arithmetic geometry (Xiao-Zhu):** the study of correspondences between mod $p$ fibers of different Shimura varieties.

# Introduction

The category $\mathrm{Rep}(G(F), \overline{\mathbb{Q}_\ell})$ needs to be modified

This can be seen from several points of view:

**Local Langlands correspondence (Vogan, Bernstein, Kaletha):** Better to study representation theory of $G(F)$ together with that of its extended pure inner forms.

**Geometric Langlands (Drinfeld, Gaitsgory):** the "local Langlands category" should be the categorical Frobenius fixed points of the category of $\overline{\mathbb{Q}_\ell}$-sheaves on the loop group of $G$.

**Arithmetic geometry (Xiao-Zhu):** the study of correspondences between mod $p$ fibers of different Shimura varieties.

***p*-adic Hodge theory (Fargus-Scholze):** moduli of $G$-bundles on the Fargues-Fontaine curve.

# Introduction

A common motif: study the representation theory $G(F)$ together with its forms $\{J_b(F)\}$ arising from the Kottwitz set $B(G)$

# Introduction

A common motif: study the representation theory $G(F)$ together with its forms $\{J_b(F)\}$ arising from the Kottwitz set $B(G)$

It turns out that the categories $\mathrm{Rep}(J_b(F), \overline{\mathbb{Q}_\ell})$ can be glued together via the category of sheaves on certain geometric objects.

# Introduction

A common motif: study the representation theory $G(F)$ together with its forms $\{J_b(F)\}$ arising from the Kottwitz set $B(G)$

It turns out that the categories $\mathrm{Rep}(J_b(F), \overline{\mathbb{Q}_\ell})$ can be glued together via the category of sheaves on certain geometric objects.

Fargues-Scholze: the categories can be glued via the moduli stack of bundles on the Fargues-Fontaine curve $X_{\mathrm{FF}}$. The set of isomorphism classes of $G$-bundles on $X_{\mathrm{FF}}$ identifies with $B(G)$ (Fargues-Fontaine, Fargues).

# Introduction

A common motif: study the representation theory $G(F)$ together with its forms $\{J_b(F)\}$ arising from the Kottwitz set $B(G)$

It turns out that the categories $\mathrm{Rep}(J_b(F), \overline{\mathbb{Q}_\ell})$ can be glued together via the category of sheaves on certain geometric objects.

Fargues-Scholze: the categories can be glued via the moduli stack of bundles on the Fargues-Fontaine curve $X_{\mathrm{FF}}$. The set of isomorphism classes of $G$-bundles on $X_{\mathrm{FF}}$ identifies with $B(G)$ (Fargues-Fontaine, Fargues).

Today: a (simpler) alternative approach using sheaves on the *Kottwitz stack*, an algebro-geometric object arising naturally from categorical considerations and the study of mod $p$ fibers of Shimura varieties.

# A Categorical Local Langlands Conjecture

Notations:

- $\mathcal{O}_F$ ring of integers of $F$, $\varpi \in \mathcal{O}_F$ a uniformizer
- Residue field $k_F = \mathcal{O}_F/\varpi$ of characteristic $p > 0$ and $|k_F| = q$.
- $\breve{F}$ the completion of the maximal unramified extension
- $\mathcal{O}_{\breve{F}}$ ring of integers, $k$ residue field (note $k = \overline{k_F}$)

# A Categorical Local Langlands Conjecture

Notations:

- $\mathcal{O}_F$ ring of integers of $F$, $\varpi \in \mathcal{O}_F$ a uniformizer
- Residue field $k_F = \mathcal{O}_F/\varpi$ of characteristic $p > 0$ and $|k_F| = q$.
- $\breve{F}$ the completion of the maximal unramified extension
- $\mathcal{O}_{\breve{F}}$ ring of integers, $k$ residue field (note $k = \overline{k_F}$)
- $\Lambda \in \{\mathbb{Q}_\ell, \mathbb{Z}_\ell, \mathbb{F}_q\}$ or any algebraic extension
- $\mathcal{G}$ be a parahoric model for $G$ over $\mathcal{O}_F$
- $\hat{G}$ the Langlands dual of $G$, as a group over $\mathbb{Z}$
- $\widetilde{F}/F$ extension splitting $G$, Galois group $\Gamma_{\widetilde{F}/F}$.

# The Kottwitz Stack

For a perfect $k_F$-algebra $R$, let

$$W_{\mathcal{O}}(R) = W(R) \otimes_{W(k_F)} \mathcal{O}_F, \quad W_{\mathcal{O},n}(R) = W_{\mathcal{O}}(R)/\varpi^n$$

denote $D_R = \operatorname{Spec} W_{\mathcal{O}}(R)$ and $D_R^* = \operatorname{Spec}(W_{\mathcal{O}}(R)[1/\varpi])$

# The Kottwitz Stack

For a perfect $k_F$-algebra $R$, let

$$W_{\mathcal{O}}(R) = W(R) \otimes_{W(k_F)} \mathcal{O}_F, \quad W_{\mathcal{O},n}(R) = W_{\mathcal{O}}(R)/\varpi^n$$

denote $D_R = \operatorname{Spec} W_{\mathcal{O}}(R)$ and $D_R^* = \operatorname{Spec}(W_{\mathcal{O}}(R)[1/\varpi])$

Recall positive loop group $L^+\mathcal{G}$ and loop group $LG$ defined by

$$L^+\mathcal{G}(R) = \mathcal{G}(W_{\mathcal{O}}(R)), \quad LG(R) = G(W_{\mathcal{O}}(R)[1/\varpi])$$

the functor $L^+\mathcal{G}$ is represented by a perfect affine scheme.

Let $\mathrm{Gr}_{\mathcal{G}} = LG/L^+\mathcal{G}$ be the associated flag variety. The functor $\mathrm{Gr}_{\mathcal{G}}$ is represented by an ind-projective perfect ind-scheme.

# The Kottwitz Stack

The *Kottwitz stack* is the prestack $\mathfrak{B}(G)$ assigning to a perfect $k_F$-algebra $R$ the groupoid

$$\mathfrak{B}(G)(R) = \left\{ (\mathcal{E}, \varphi) \middle| \, \mathcal{E} \text{ is an étale } G\text{-torsor on } D_R^*, \; \varphi : \mathcal{E} \simeq \sigma_R^* \mathcal{E} \right\}$$

where $\mathcal{E}$ is a $G$-torsor over $D_R^*$, which can be trivialized over $D_{R'}^*$ for some étale covering map $R \to R'$,

Equivalently,

$$\mathfrak{B}(G) = (LG/\mathrm{Ad}_\sigma LG)_{\acute{e}t},$$

i.e. the étale sheafification of the prestack quotient of $LG$ by $\mathrm{Ad}_\sigma$-conjugation by $LG$.

# The Newton Stratification

For every separably closed field $K$ over $k_F$ set of isomorphism classes of the groupoid $\mathfrak{B}(G)(K)$ identifies with the Kottwitz set.

# The Newton Stratification

For every separably closed field $K$ over $k_F$ set of isomorphism classes of the groupoid $\mathfrak{B}(G)(K)$ identifies with the Kottwitz set.

$\mathfrak{B}(G)$ is stratified by the poset $B(G)$, for an element $b$ define

$$\mathfrak{B}(G)_{\leq b}(R) = \left\{ (\mathcal{E}, \varphi) \in \mathfrak{B}(G)(R) \middle| \; b_x := (\mathcal{E}_x, \varphi_x) \leq b, \; x \in \operatorname{Spec} R, \right\}$$

Results of Rapoport-Richartz imply that the inclusion $\mathfrak{B}(G)_{\leq b} \subset \mathfrak{B}(G)$ is a finitely presented closed embedding.

# The Newton Stratification

For every separably closed field $K$ over $k_F$ set of isomorphism classes of the groupoid $\mathfrak{B}(G)(K)$ identifies with the Kottwitz set.

$\mathfrak{B}(G)$ is stratified by the poset $B(G)$, for an element $b$ define

$$\mathfrak{B}(G)_{\leq b}(R) = \left\{(\mathcal{E}, \varphi) \in \mathfrak{B}(G)(R) \middle|\ b_x := (\mathcal{E}_x, \varphi_x) \leq b,\ x \in \operatorname{Spec} R, \right\}$$

Results of Rapoport-Richartz imply that the inclusion $\mathfrak{B}(G)_{\leq b} \subset \mathfrak{B}(G)$ is a finitely presented closed embedding.

For $b \in B(G)$ set

$$\mathfrak{B}(G)_b = \mathfrak{B}(G)_b \setminus \cup_{b' < b} \mathfrak{B}(G)_{b'}$$

$\mathfrak{B}(G)_b \subset \mathfrak{B}(G)_{\leq b}$ is a finitely presented affine open embedding (follows from results of Vasiu, Hartl-Viehmann)

# Sheaves on the Kottwitz Stack

We can make sense of the category of $\Lambda$-sheaves on objects like $\mathfrak{B}(G)$.

# Sheaves on the Kottwitz Stack

We can make sense of the category of $\Lambda$-sheaves on objects like $\mathfrak{B}(G)$.

For a perfect ring $R$ with presentation $R = \mathrm{colim}_i R_i$ as a colimit of (perfectly) finite type $k$-algebras

$$\mathrm{Shv}(\mathrm{Spec}\, R, \Lambda) = \varinjlim_i \mathrm{Ind}\big(\mathrm{D}_{\mathrm{ctf}}(\mathrm{Spec}\, R_i, \Lambda)\big)$$

transition maps are given by !-pullbacks.

# Sheaves on the Kottwitz Stack

We can make sense of the category of $\Lambda$-sheaves on objects like $\mathfrak{B}(G)$.

For a perfect ring $R$ with presentation $R = \mathrm{colim}_i R_i$ as a colimit of (perfectly) finite type $k$-algebras

$$\mathrm{Shv}(\mathrm{Spec}\, R, \Lambda) = \varinjlim_i \mathrm{Ind}\big(\mathrm{D}_{\mathrm{ctf}}(\mathrm{Spec}\, R_i, \Lambda)\big)$$

transition maps are given by !-pullbacks.For example,

$$\mathrm{Shv}(L^+\mathcal{G}, \Lambda) = \varinjlim_n \mathrm{Shv}(L^n\mathcal{G}, \Lambda),$$

where $L^n\mathcal{G}$ is the partial jet scheme $L^n\mathcal{G}(R) = \mathcal{G}(W_{\mathcal{O},n}(R))$.

# Sheaves on the Kottwitz Stack

For every functor $\mathcal{X} \colon \mathrm{Aff}_k^{\mathrm{pf}} \to \mathrm{Ani}$ (also called a prestack)

$$\mathrm{Shv}(\mathcal{X}, \Lambda) = \lim_{\mathrm{Spec}\, R \to \mathcal{X}} \mathrm{Shv}(\mathrm{Spec}\, R, \Lambda)$$

A natural transformation $f \colon \mathcal{X} \to \mathcal{Y}$ induces a functor

$$f^! \colon \mathrm{Shv}(\mathcal{Y}, \Lambda) \to \mathrm{Shv}(\mathcal{X}, \Lambda)$$

For specific types of prestacks $\mathcal{X}, \mathcal{Y}$ and morphisms $f$, one can also define other functors like $f_!, f^*, f_*$ satisfying expected properties.

In particular, we have categories $\mathrm{Shv}(\mathfrak{B}(G), \Lambda)$ and $\mathrm{Shv}(\mathfrak{B}(G)_b, \Lambda)$ for every $b \in B(G)$.

# Sheaves on the Kottwitz Stack

As in the case of schemes, there are adjunctions

$$\begin{tikzcd}[column sep=large] \mathrm{Shv}(\mathfrak{B}(G)_b, \Lambda) \arrow[r, hook, bend left, "j_!"] \arrow[r, hook, bend right, "j_*"'] & \mathrm{Shv}(\mathfrak{B}(G)_{\leq b}, \Lambda) \arrow[l, "j^*"'] \arrow[r, bend left, "i^*"] \arrow[r, bend right, "i^!"'] & \mathrm{Shv}(\mathfrak{B}(G)_{< b}, \Lambda) \arrow[l, hook, "i_*"'] \end{tikzcd}$$

# Sheaves on the Kottwitz Stack

As in the case of schemes, there are adjunctions

$$\begin{tikzcd} \mathrm{Shv}(\mathfrak{B}(G)_b, \Lambda) \arrow[r, hook, bend left, "j_!"] \arrow[r, hook, bend right, "j_*"'] & \mathrm{Shv}(\mathfrak{B}(G)_{\le b}, \Lambda) \arrow[l, "j^*"] \arrow[r, bend left, "i^*"] \arrow[r, bend right, "i^!"'] & \mathrm{Shv}(\mathfrak{B}(G)_{<b}, \Lambda) \arrow[l, hook, "i_*"] \end{tikzcd}$$

For $\mathcal{F} \in \mathrm{Shv}(\mathfrak{B}(G)_{\le b}, \Lambda)$ there are natural fiber sequences

$$i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}$$
$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}$$

inducing a semi-orthogonal decomposition of $\mathrm{Shv}(\mathfrak{B}(G), \Lambda)$ in terms of the categories $\mathrm{Shv}(\mathfrak{B}(G)_b, \Lambda)$

When $b \in B(G)$ is basic, $\mathfrak{B}(G)_b = \mathfrak{B}(G)_{\le b}$.

# Sheaves on the Kottwitz Stack

## Theorem (H.-Zhu)

1. For every $b \in B(G)$ there is a canonical equivalence

$$\mathrm{Rep}(J_b(F), \Lambda) \simeq \mathrm{Shv}(\mathfrak{B}(G)_b, \Lambda)$$

2. The category $\mathrm{Shv}(\mathfrak{B}(G), \Lambda)$ is compactly generated. An object is compact if and only if the restriction to each Newton stratum is compact.

# Sheaves on the Kottwitz Stack

## Theorem (H.-Zhu)

1. For every $b \in B(G)$ there is a canonical equivalence

$$\mathrm{Rep}(J_b(F), \Lambda) \simeq \mathrm{Shv}(\mathfrak{B}(G)_b, \Lambda)$$

2. The category $\mathrm{Shv}(\mathfrak{B}(G), \Lambda)$ is compactly generated. An object is compact if and only if the restriction to each Newton stratum is compact.
3. There is a self-duality $\mathrm{Shv}(\mathfrak{B}(G), \Lambda)$

$$\mathbb{D}^{\mathrm{coh}} \colon \mathrm{Shv}(\mathfrak{B}(G), \Lambda)^{\omega} \simeq (\mathrm{Shv}(\mathfrak{B}(G), \Lambda)^{\omega})^{op}$$

# Sheaves on the Kottwitz Stack

## Theorem (H.-Zhu)

1. For every $b \in B(G)$ there is a canonical equivalence
$$\mathrm{Rep}(J_b(F), \Lambda) \simeq \mathrm{Shv}(\mathfrak{B}(G)_b, \Lambda)$$
2. The category $\mathrm{Shv}(\mathfrak{B}(G), \Lambda)$ is compactly generated. An object is compact if and only if the restriction to each Newton stratum is compact.
3. There is a self-duality $\mathrm{Shv}(\mathfrak{B}(G), \Lambda)$
$$\mathbb{D}^{\mathrm{coh}} \colon \mathrm{Shv}(\mathfrak{B}(G), \Lambda)^{\omega} \simeq (\mathrm{Shv}(\mathfrak{B}(G), \Lambda)^{\omega})^{op}$$
such that for every $b \in B(G)$
$$\mathbb{D}^{\mathrm{coh}} i_{b,*} \simeq i_{b,!} \mathbb{D}^{\mathrm{coh}}_{\mathrm{Rep}(J_b(F),\Lambda)}[-\langle 2\rho, \nu_b \rangle]$$

# The Stack of Langlands Parameters

Consider the $C$-group ${}^cG = \hat{G} \rtimes (\mathbb{G}_m \times \Gamma_{\widetilde{F}/F})$ as a group scheme over $\mathbb{Z}$.

There is a stack $\mathrm{Loc}_{{}^cG,F}$ over $\mathbb{Z}_\ell$ classifying families of Langlands parameters considered independently by Dat-Helm-Kurinczuk-Moss, Scholze, and Zhu.

$$\mathrm{Loc}^{\square}_{{}^cG,F}(A) = \{\text{continuous } \rho : W_F \to {}^cG(A)\}, \ \mathrm{Loc}_{{}^cG,F} := \mathrm{Loc}^{\square}_{{}^cG,F}/\hat{G}$$

$\mathrm{Loc}_{{}^cG,F}$ is representable, reduced, flat and locally of finite presentation over $\mathbb{Z}_\ell$, and a local complete intersection.

We denote by $\mathrm{Loc}_{{}^cG,F,\Lambda}$ the base change of $\mathrm{Loc}_{{}^cG,F}$ to $\Lambda$.

# The Stack of Langlands Parameters

Assume $\widetilde{F}/F$ is tamely ramified. In this case there is an open and closed substack $\mathrm{Loc}^{\mathrm{tame}}_{{}^cG,F} \subseteq \mathrm{Loc}_{{}^cG,F}$ classifying parameters whose kernel includes the wild inertia subgroup.

Fix a topological generator $\tau$ of tame inertia and fix a Frobenius $\sigma$.

We have a presentation:

$$\mathrm{Loc}^{\mathrm{tame}}_{{}^cG,F} \simeq \left\{(g,h) \in \hat{G}\tau \times \hat{G}\sigma \subset {}^cG \times {}^cG \mid hgh^{-1} = g^q\right\} / \hat{G}$$

# The Stack of Langlands Parameters

The tangent complex at a point $\rho$ of $\mathrm{Loc}_{^cG}$ is quasi-isomorphic to the continuous cohomology complex $C^*_{\mathrm{cts}}(W_F, \mathrm{Ad}_\rho)[1]$.

# The Stack of Langlands Parameters

The tangent complex at a point $\rho$ of $\mathrm{Loc}_{{}^cG}$ is quasi-isomorphic to the continuous cohomology complex $C^*_{\mathrm{cts}}(W_F, \mathrm{Ad}_\rho)[1]$.

In particular,

$$\mathrm{Sing}(\mathrm{Loc}_{{}^cG,F}) = \left\{(\rho,\xi) \;\middle|\; \rho \in \mathrm{Loc}_{{}^cG,F},\ \xi \in H_2(W_F, \mathrm{Ad}^*_\rho)\right\}$$

By Tate duality, $H_2(W_F, \mathrm{Ad}^*_\rho) \cong (\hat{\mathfrak{g}}^*)^{\rho|_{I_F}=1, \rho(\sigma)=q^{-1}}$.

# The Stack of Langlands Parameters

The tangent complex at a point $\rho$ of $\mathrm{Loc}_{^cG}$ is quasi-isomorphic to the continuous cohomology complex $C^*_{\mathrm{cts}}(W_F, \mathrm{Ad}_\rho)[1]$.

In particular,

$$\mathrm{Sing}(\mathrm{Loc}_{^cG,F}) = \left\{(\rho,\xi) \;\middle|\; \rho \in \mathrm{Loc}_{^cG,F},\ \xi \in H_2(W_F, \mathrm{Ad}^*_\rho)\right\}$$

By Tate duality, $H_2(W_F, \mathrm{Ad}^*_\rho) \cong (\hat{\mathfrak{g}}^*)^{\rho|_{I_F}=1, \rho(\sigma)=q^{-1}}$.
Let $\hat{\mathcal{N}}^* \subseteq \hat{\mathfrak{g}}^*$ be the nilpotent cone. We can then define

$$\hat{\mathcal{N}}_{^cG,F} = \left\{(\rho,\xi) \in \mathrm{Sing}(\mathrm{Loc}_{^cG,F}) \;\middle|\; \rho \in \mathrm{Loc}_{^cG,F},\ \xi \in \hat{\mathcal{N}}^*\right\}$$

# Categorical Arithmetic Local Langlands Correspondence

## Conjecture (Zhu)

Assume $G$ is quasi-split and fix Whittaker data $(U, \psi)$. There is a canonical equivalence of $\infty$-categories

$$\mathbb{L}_G \colon \mathrm{Ind}\big(\mathrm{Coh}_{\hat{\mathcal{N}}_{^cG,F,\Lambda}}(\mathrm{Loc}_{^cG,F,\Lambda})\big) \simeq \mathrm{Shv}(\mathfrak{B}(G), \Lambda)$$

Which is compatible with parabolic induction, intertwines cohomological duality and Serre duality (up to Cartan involution), and sends the object $\mathcal{O}_{\mathrm{Loc}_{^cG,F}}$ to the object $i_{e,*}\big(c\text{-}\mathrm{ind}_{U(F)}^{G(F)}\psi\big)$

# Categorical Arithmetic Local Langlands Correspondence

## Conjecture (Zhu)

Assume $G$ is quasi-split and fix Whittaker data $(U, \psi)$. There is a canonical equivalence of $\infty$-categories

$$\mathbb{L}_G \colon \mathrm{Ind}\big(\mathrm{Coh}_{\hat{\mathcal{N}}_{^cG,F,\Lambda}}(\mathrm{Loc}_{^cG,F,\Lambda})\big) \simeq \mathrm{Shv}(\mathfrak{B}(G), \Lambda)$$

Which is compatible with parabolic induction, intertwines cohomological duality and Serre duality (up to Cartan involution), and sends the object $\mathcal{O}_{\mathrm{Loc}_{^cG,F}}$ to the object $i_{e,*}\big(c\text{-}\mathrm{ind}_{U(F)}^{G(F)}\psi\big)$

- Analogous to the conjecture made by Fargues-Scholze
- If $\Lambda$ is of characteristic zero, $\hat{\mathcal{N}}_{^cG,F} = \mathrm{Sing}(\mathrm{Loc}_{^cG,F,\Lambda})$
- Follows from class field theory for unramified tori

# The Unipotent Langlands Category

Assume $G$ is unramified, i.e. that the extension $\widetilde{F}/F$ is unramified. We fix a pinning $(G, B, T, e)$.

# The Unipotent Langlands Category

Assume $G$ is unramified, i.e. that the extension $\widetilde{F}/F$ is unramified. We fix a pinning $(G, B, T, e)$.

Consider the full subcategory

$$\mathrm{Shv}^{\mathrm{unip}}(\mathfrak{B}(G), \overline{\mathbb{Q}_\ell}) \subseteq \mathrm{Shv}(\mathfrak{B}(G), \overline{\mathbb{Q}_\ell})$$

consisting of objects $\mathcal{F} \in \mathrm{Shv}(\mathfrak{B}(G), \overline{\mathbb{Q}_\ell})$ such that for all $b \in B(G)$, the cohomologies of the complex

$$i_b^!(\mathcal{F}) \in \mathrm{Rep}(J_b(F), \overline{\mathbb{Q}_\ell})$$

are unipotent in the sense of Lusztig.

By duality, this is equivalent to the same condition for $i_b^*(\mathcal{F})$.

# The Stack of Unipotent Langlands Parameters

On the other hand, consider the substack $\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F} \subseteq \mathrm{Loc}^{\mathrm{tame}}_{{}^cG,F}$ classifying representations which factor through the tame quotient of $W_F$ and carry unipotent monodromy,

$$\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F} \simeq \left\{(g,h) \in \hat{\mathcal{U}} \times \hat{G}\sigma \subset {}^cG \times {}^cG \mid hgh^{-1} = g^q\right\} / \hat{G}$$

for $\Lambda$ of characteristic zero, $\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F,\Lambda} \subseteq \mathrm{Loc}^{\mathrm{tame}}_{{}^cG,F,\Lambda}$ is open and closed.

# The Stack of Unipotent Langlands Parameters

On the other hand, consider the substack $\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F} \subseteq \mathrm{Loc}^{\mathrm{tame}}_{{}^cG,F}$ classifying representations which factor through the tame quotient of $W_F$ and carry unipotent monodromy,

$$\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F} \simeq \left\{(g,h) \in \hat{\mathcal{U}} \times \hat{G}\sigma \subset {}^cG \times {}^cG \mid hgh^{-1} = g^q\right\} / \hat{G}$$

for $\Lambda$ of characteristic zero, $\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F,\Lambda} \subseteq \mathrm{Loc}^{\mathrm{tame}}_{{}^cG,F,\Lambda}$ is open and closed.

Consider the map $\pi^{\mathrm{unip}} \colon \mathrm{Loc}^{\mathrm{unip}}_{{}^cB,F,\Lambda} \to \mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F,\Lambda}$. The object

$$\mathrm{CohSpr}^{\mathrm{unip}}_{{}^cG,F,\Lambda} = \pi^{\mathrm{unip}}_* \mathcal{O}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cB,F,\Lambda}}$$

is called the *coherent Springer sheaf*

# Unipotent Arithmetic Local Langlands Correspondence

## Theorem (H.-Zhu)

There is a canonical equivalence of $\infty$-categories

$$\mathbb{L}_G\colon \mathrm{Ind}(\mathrm{Coh}(\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F,\overline{\mathbb{Q}_\ell}})) \simeq \mathrm{Shv}^{\mathrm{unip}}(\mathfrak{B}(G), \overline{\mathbb{Q}_\ell})$$

Such that for $I \subset K \subset G(F)$ determined by the pinning:

$$\mathbb{L}_G(\mathrm{CohSpr}^{\mathrm{unip}}_{{}^cG,F}) \simeq i_{e,*}(C_c^\infty(G(F)/I, \overline{\mathbb{Q}_\ell}))$$
$$\mathbb{L}_G(\mathcal{O}_{\mathrm{Loc}^{\mathrm{unr}}_{{}^cG,F}}) = i_{e,*}(C_c^\infty(G(F)/K, \overline{\mathbb{Q}_\ell}))$$
$$\mathbb{L}_G(\mathcal{O}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F}}) \simeq i_{e,*}\left(C_c^\infty(G(F)/I, \overline{\mathbb{Q}_\ell}) \otimes_{H_I} (C_c^\infty(I\backslash G(F)/I^u)\right)$$

# Unipotent Arithmetic Local Langlands Correspondence

**Theorem (H.-Zhu)**

There is a canonical equivalence of $\infty$-categories

$$\mathbb{L}_G : \mathrm{Ind}(\mathrm{Coh}(\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F,\overline{\mathbb{Q}_\ell}})) \simeq \mathrm{Shv}^{\mathrm{unip}}(\mathfrak{B}(G), \overline{\mathbb{Q}_\ell})$$

Such that for $I \subset K \subset G(F)$ determined by the pinning:

$$\mathbb{L}_G(\mathrm{CohSpr}^{\mathrm{unip}}_{{}^cG,F}) \simeq i_{e,*}(C_c^\infty(G(F)/I, \overline{\mathbb{Q}_\ell}))$$
$$\mathbb{L}_G(\mathcal{O}_{\mathrm{Loc}^{\mathrm{unr}}_{{}^cG,F}}) = i_{e,*}(C_c^\infty(G(F)/K, \overline{\mathbb{Q}_\ell}))$$
$$\mathbb{L}_G(\mathcal{O}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F}}) \simeq i_{e,*}\big(C_c^\infty(G(F)/I, \overline{\mathbb{Q}_\ell}) \otimes_{H_I} (C_c^\infty(I\backslash G(F)/I^u)\big)$$

- Related results of Ben-Zvi-Chen-Helm-Nadler by a different method
- Generalizes to $\overline{\mathbb{F}_\ell}$ once Bezrukavnikov's equivalence is available

# Unipotent Arithmetic Local Langlands Correspondence

## Corollary

There are canonical equivalences

$$H_I \simeq \mathrm{End}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F,\overline{\mathbb{Q}_\ell}}}(\mathrm{CohSpr}^{\mathrm{unip}}_{{}^cG,F}), \quad H_K \simeq \mathrm{End}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F,\overline{\mathbb{Q}_\ell}}}(\mathcal{O}_{\mathrm{Loc}^{\mathrm{unr}}_{{}^cG,F}})$$

# Unipotent Arithmetic Local Langlands Correspondence

## Corollary

There are canonical equivalences

$$H_I \simeq \mathrm{End}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F,\overline{\mathbb{Q}_\ell}}}(\mathrm{CohSpr}^{\mathrm{unip}}_{{}^cG,F}), \quad H_K \simeq \mathrm{End}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F,\overline{\mathbb{Q}_\ell}}}(\mathcal{O}_{\mathrm{Loc}^{\mathrm{unr}}_{{}^cG,F}})$$

For every basic $b \in B(G)$ let $I_b \subseteq J_b(F)$ denote the corresponding Iwahori. The theorem implies the existence of a coherent sheaf $\mathfrak{A}_b$ such that

$$\mathbb{L}_G(\mathfrak{A}_b) = (i_b)_*(C_c^\infty(J_b(F)/I_b, \overline{\mathbb{Q}_\ell}))$$

and consequently,

$$H_{I_b} \simeq \mathrm{End}_{\mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F,\overline{\mathbb{Q}_\ell}}}(\mathfrak{A}_b).$$

The sheaves $\mathfrak{A}_b$ can be explicitly described.

# The Stack of Local Shtukas

Recall the stack of local shtukas

$$\mathrm{Sht}^{\mathrm{loc}}(R) = \left\{(\mathcal{E}, \varphi) \mid \mathcal{E} \text{ is a } \mathcal{G}\text{-torsors on } D_R,\ \varphi : \mathcal{E} \dashrightarrow \sigma_R^* \mathcal{E}\right\}$$

Studied by Genestier-Lafforgue, Hartl-Viehmann, Xiao-Zhu.

# The Stack of Local Shtukas

Recall the stack of local shtukas

$$\mathrm{Sht}^{\mathrm{loc}}(R) = \left\{(\mathcal{E}, \varphi) \mid \mathcal{E} \text{ is a } \mathcal{G}\text{-torsors on } D_R,\ \varphi : \mathcal{E} \dashrightarrow \sigma_R^* \mathcal{E}\right\}$$

Studied by Genestier-Lafforgue, Hartl-Viehmann, Xiao-Zhu.

There is an isomorphism

$$\mathrm{Sht}^{\mathrm{loc}} \cong \left(LG/\mathrm{Ad}_\sigma L^+\mathcal{G}\right)_{\acute{e}t}$$

# The Stack of Local Shtukas

Recall the stack of local shtukas

$$\mathrm{Sht}^{\mathrm{loc}}(R) = \left\{(\mathcal{E}, \varphi) \mid \mathcal{E} \text{ is a } \mathcal{G}\text{-torsors on } D_R,\ \varphi : \mathcal{E} \dashrightarrow \sigma_R^* \mathcal{E}\right\}$$

Studied by Genestier-Lafforgue, Hartl-Viehmann, Xiao-Zhu.

There is an isomorphism

$$\mathrm{Sht}^{\mathrm{loc}} \cong \left(LG/\mathrm{Ad}_\sigma L^+\mathcal{G}\right)_{\acute{e}t}$$

Restriction of torsors from $D_R$ to $D_R^*$ gives the *Newton map*

$$\mathrm{Nt} : \mathrm{Sht}^{\mathrm{loc}} \to \mathfrak{B}(G)$$

Key fact: $\mathrm{Nt}$ is ind-proper, looks like a fibration with fiber $\mathrm{Gr}_{\mathcal{G}}$.

# Local Shtukas and the Kottwitz Stack

More generally, we have the $n$-th iterated local Hecke stack:

$$\mathrm{Hk}_n(\mathrm{Sht}^{\mathrm{loc}})(R) = \left\{ \mathcal{E}_0 \dashrightarrow \cdots \dashrightarrow \mathcal{E}_n \dashrightarrow {}^{\sigma}\mathcal{E}_0 \,\middle|\, \mathcal{E}_i \text{ a } \mathcal{G}\text{-torsor on } D_R. \right\}$$

# Local Shtukas and the Kottwitz Stack

More generally, we have the $n$-th iterated local Hecke stack:

$$\mathrm{Hk}_n(\mathrm{Sht}^{\mathrm{loc}})(R) = \left\{ \mathcal{E}_0 \dashrightarrow \cdots \dashrightarrow \mathcal{E}_n \dashrightarrow {}^{\sigma}\mathcal{E}_0 \,\middle|\, \mathcal{E}_i \text{ a } \mathcal{G}\text{-torsor on } D_R. \right\}$$

There is an equivalence $\mathrm{Hk}_n \cong (LG^{n+1}/L^+\mathcal{G}^{n+1})_{ét}$ with action

$$(k_0, k_1, \ldots, k_n) \cdot (g_0, g_1, \ldots, g_n) = (k_0 g_0 k_1^{-1}, k_1 g_2 k_2^{-1}, \ldots, k_n g_n \sigma(k_0)^{-1}),$$

More generally, we have the $n$-th iterated local Hecke stack:

$$\mathrm{Hk}_n(\mathrm{Sht}^{\mathrm{loc}})(R) = \left\{ \mathcal{E}_0 \dashrightarrow \cdots \dashrightarrow \mathcal{E}_n \dashrightarrow {}^{\sigma}\mathcal{E}_0 \,\middle|\, \mathcal{E}_i \text{ a } \mathcal{G}\text{-torsor on } D_R. \right\}$$

There is an equivalence $\mathrm{Hk}_n \cong (LG^{n+1}/L^+\mathcal{G}^{n+1})_{\acute{e}t}$ with action

$$(k_0, k_1, \ldots, k_n) \cdot (g_0, g_1, \ldots, g_n) = (k_0 g_0 k_1^{-1}, k_1 g_2 k_2^{-1}, \ldots, k_n g_n \sigma(k_0)^{-1}),$$

Consider the simplicial object $\mathrm{Hk}_\bullet(\mathrm{Sht}^{\mathrm{loc}})$

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathrm{Hk}_1(\mathrm{Sht}^{\mathrm{loc}}) \;\substack{\longrightarrow\\\longrightarrow}\; \mathrm{Sht}^{\mathrm{loc}} \xrightarrow{\mathrm{Nt}} \mathfrak{B}(G)$$

$$d_i(g_0, \ldots, g_{i-1}, g_i, \ldots g_n) = \begin{cases} (g_0, \ldots g_{i-1}g_i, \ldots, g_n), & i \neq 0 \\ (g_1, g_2, \ldots, g_n\sigma(g_0)), & i = 0 \end{cases}$$

$\mathrm{Hk}_\bullet(\mathrm{Sht}^{\mathrm{loc}})$ is isomorphic to Čech nerve of the Newton map

$$\mathrm{Nt} : \mathrm{Sht}^{\mathrm{loc}} \to \mathfrak{B}(G)$$

## Local Shtukas and the Kottwitz Stack

$\mathrm{Hk}_\bullet(\mathrm{Sht}^{\mathrm{loc}})$ is isomorphic to Čech nerve of the Newton map

$$\mathrm{Nt} : \mathrm{Sht}^{\mathrm{loc}} \to \mathfrak{B}(G)$$

Ind-proper descent gives a colimit of presentable $\infty$-categories

$$\cdots \mathrm{Shv}(\mathrm{Hk}_1(\mathrm{Sht}^{\mathrm{loc}}), \Lambda) \rightrightarrows \mathrm{Shv}(\mathrm{Sht}^{\mathrm{loc}}, \Lambda) \xrightarrow{\mathrm{Nt}_!} \mathrm{Shv}(\mathfrak{B}(G), \Lambda)$$

# Local Shtukas and the Kottwitz Stack

$\mathrm{Hk}_\bullet(\mathrm{Sht}^{\mathrm{loc}})$ is isomorphic to Čech nerve of the Newton map

$$\mathrm{Nt} : \mathrm{Sht}^{\mathrm{loc}} \to \mathfrak{B}(G)$$

Ind-proper descent gives a colimit of presentable $\infty$-categories

$$\cdots \mathrm{Shv}(\mathrm{Hk}_1(\mathrm{Sht}^{\mathrm{loc}}), \Lambda) \rightrightarrows \mathrm{Shv}(\mathrm{Sht}^{\mathrm{loc}}, \Lambda) \xrightarrow{\mathrm{Nt}_!} \mathrm{Shv}(\mathfrak{B}(G), \Lambda)$$

There is a Verdier duality on the ind-stacks $\mathrm{Hk}_n(\mathrm{Sht}^{\mathrm{loc}})$, subject to a choice of dimension theory. Compatible choices of dimensions induce the duality

$$\mathbb{D}^{\mathrm{coh}} : \mathrm{Shv}(\mathfrak{B}(G), \Lambda) \xrightarrow{\sim} \mathrm{Shv}(\mathfrak{B}(G), \Lambda)^\vee$$

# Local Shtukas and the Kottwitz Stack

Now fix the following situation:

- $\mathcal{G}$ is the smooth model over $\mathcal{O}_F$ corresponding to $I$.
- Denote $L^+\mathcal{G}$ by $\mathcal{I}$ so that $\mathcal{I}(k_F) = I$.
- We denote $\mathrm{Gr}_{\mathcal{I}}$ by $\mathrm{Fl}$, $\pi \colon LG \to \mathrm{Fl}$ the projection
- $\widetilde{W}$ the Iwahori-Weyl group, $W \subseteq \widetilde{W}$ finite Weyl group.

# Local Shtukas and the Kottwitz Stack

Now fix the following situation:

- $\mathcal{G}$ is the smooth model over $\mathcal{O}_F$ corresponding to $I$.
- Denote $L^+\mathcal{G}$ by $\mathcal{I}$ so that $\mathcal{I}(k_F) = I$.
- We denote $\mathrm{Gr}_{\mathcal{I}}$ by $\mathrm{Fl}$, $\pi\colon LG \to \mathrm{Fl}$ the projection
- $\widetilde{W}$ the Iwahori-Weyl group, $W \subseteq \widetilde{W}$ finite Weyl group.

Decomposition into affine Schubert cells

$$\mathrm{Fl} = \bigcup_{w \in \widetilde{W}} \mathrm{Fl}_w, \; LG = \bigcup_{w \in \widetilde{W}} LG_w, \; \mathrm{Sht}^{\mathrm{loc}} = \bigcup_{w \in \widetilde{W}} \mathrm{Sht}^{\mathrm{loc}}_w$$

with $LG_w = \pi^{-1}(\mathrm{Fl}_w)$ and $\mathrm{Sht}^{\mathrm{loc}}_w \simeq (LG_w/\mathrm{Ad}_\sigma \mathcal{I})_{ét}$

# Local Shtukas and the Kottwitz Stack

Denote $i_w \colon \mathrm{Sht}_w^{\mathrm{loc}} \to \mathrm{Sht}^{\mathrm{loc}}$. For $w = 1$, by Lang's theorem

$$\mathrm{Sht}_1^{\mathrm{loc}} \simeq \left(\mathcal{I}/\mathrm{Ad}_\sigma \mathcal{I}\right)_{\acute{e}t} \simeq \left(\mathcal{I}/\mathrm{Ad}_\sigma \mathcal{I}\right)_{fpqc} \simeq \left(\operatorname{Spec} k/I\right)_{fpqc}$$

# Local Shtukas and the Kottwitz Stack

Denote $i_w \colon \mathrm{Sht}_w^{\mathrm{loc}} \to \mathrm{Sht}^{\mathrm{loc}}$. For $w = 1$, by Lang's theorem

$$\mathrm{Sht}_1^{\mathrm{loc}} \simeq \big(\mathcal{I}/\mathrm{Ad}_\sigma\mathcal{I}\big)_{\acute{e}t} \simeq \big(\mathcal{I}/\mathrm{Ad}_\sigma\mathcal{I}\big)_{fpqc} \simeq \big(\operatorname{Spec} k/I\big)_{fpqc}$$

Fits intro a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}(I,\Lambda) \arrow[r, "i_{e,*}"] \arrow[d, "c\text{-}\mathrm{ind}_I^{G(F)}"] & \mathrm{Shv}(\mathrm{Sht}^{\mathrm{loc}},\Lambda) \arrow[d, "\mathrm{Nt}_!"] \\
\mathrm{Rep}(G(F),\Lambda) \arrow[r, "i_{e,*}"] & \mathrm{Shv}(\mathfrak{B}(G),\Lambda)
\end{tikzcd}$$

In particular

$$\mathrm{Nt}_!(i_{e,*}\Lambda) \simeq i_{e,*}c\text{-}\mathrm{ind}_I^{G(F)}(\Lambda)$$

# Local Shtukas and the Kottwitz Stack

Denote $i_w\colon \mathrm{Sht}^{\mathrm{loc}}_w \to \mathrm{Sht}^{\mathrm{loc}}$. For $w = 1$, by Lang's theorem

$$\mathrm{Sht}^{\mathrm{loc}}_1 \simeq \left(\mathcal{I}/\mathrm{Ad}_\sigma \mathcal{I}\right)_{\acute{e}t} \simeq \left(\mathcal{I}/\mathrm{Ad}_\sigma \mathcal{I}\right)_{fpqc} \simeq \left(\operatorname{Spec} k/I\right)_{fpqc}$$

Fits intro a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}(I,\Lambda) \arrow[r, "i_{e,*}"] \arrow[d, "c\text{-}\mathrm{ind}_I^{G(F)}"] & \mathrm{Shv}(\mathrm{Sht}^{\mathrm{loc}},\Lambda) \arrow[d, "\mathrm{Nt}_!"] \\
\mathrm{Rep}(G(F),\Lambda) \arrow[r, "i_{e,*}"] & \mathrm{Shv}(\mathfrak{B}(G),\Lambda)
\end{tikzcd}$$

In particular

$$\mathrm{Nt}_!(i_{e,*}\Lambda) \simeq i_{e,*}c\text{-}\mathrm{ind}_I^{G(F)}(\Lambda)$$

This can be generalized to any $b \in B(G)$

# Local Shtukas and the Kottwitz Stack

$w \in \widetilde{W}$ is called $\sigma$-straight if $\ell(w\sigma(w)\cdots\sigma^{n-1}(w)) = n\ell(w)$., $n > 0$
A $\sigma$-conjugacy class of $\widetilde{W}$ is *straight* if it contains a $\sigma$-straight element.

# Local Shtukas and the Kottwitz Stack

$w \in \widetilde{W}$ is called $\sigma$-straight if $\ell(w\sigma(w)\cdots\sigma^{n-1}(w)) = n\ell(w).$, $n > 0$
A $\sigma$-conjugacy class of $\widetilde{W}$ is *straight* if it contains a $\sigma$-straight element.

**Theorem (X. He)**

$$\left\{\text{straight } \sigma\text{-conjugacy classes in } \widetilde{W}\right\} \longleftrightarrow B(G).$$

# Local Shtukas and the Kottwitz Stack

$w \in \widetilde{W}$ is called $\sigma$-straight if $\ell(w\sigma(w)\cdots\sigma^{n-1}(w)) = n\ell(w).,\ n > 0$
A $\sigma$-conjugacy class of $\widetilde{W}$ is *straight* if it contains a $\sigma$-straight element.

**Theorem (X. He)**

$$\left\{\text{straight } \sigma\text{-conjugacy classes in } \widetilde{W}\right\} \longleftrightarrow B(G).$$

If $w_b$ is $\sigma$-straight corresponding to $b$ then $\mathrm{Sht}^{\mathrm{loc}}_{w_b} \cong \left(\mathrm{Spec}\, k/I_b\right)_{fpqc}$

$$\begin{tikzcd}
\mathrm{Rep}(I_b, \Lambda) \arrow[r, "i_{w,*}"] \arrow[d, "c\text{-}\mathrm{ind}_{I_b}^{J_b(F)}"] & \mathrm{Shv}(\mathrm{Sht}^{\mathrm{loc}}, \Lambda) \arrow[d, "\mathrm{Nt}_!"] \\
\mathrm{Rep}(J_b(F), \Lambda) \arrow[r, "i_{b,*}"] & \mathrm{Shv}(\mathfrak{B}(G), \Lambda)
\end{tikzcd},$$

and similarly with $i_{b,!}$ instead of $i_{b,*}$.

# Unipotent Arithmetic Local Langlands Correspondence

**Theorem (Bezrukavnikov)**

There is a canonical equivalence of monoidal categories

$$\mathbb{B}\colon \mathrm{Ind}\big(\mathrm{Coh}(\mathrm{St}^{\mathrm{unip}}_{\hat{G}}/\hat{G})\big) \cong \mathrm{Shv}(\mathcal{I}\backslash LG/\mathcal{I}, \overline{\mathbb{Q}}_\ell)$$

- $\widetilde{\mathcal{U}}_{\hat{G}} \to \hat{G}$ the (multiplicative) Springer resolution
- $\mathrm{St}^{\mathrm{unip}}_{\hat{G}} = \widetilde{\mathcal{U}}_{\hat{G}} \times^{L}_{\hat{G}} \widetilde{\mathcal{U}}_{\hat{G}}$ is the unipotent Steinberg variety
- Intertwines the pullback of Frobenius on $\mathcal{I}\backslash LG/\mathcal{I}$ with the pullback of the map on $\mathrm{St}^{\mathrm{unip}}_{\hat{G}}$ induced by the map $g \mapsto \sigma^{-1}(g^q)$ on $\hat{G}$.

# Unipotent Arithmetic Local Langlands Correspondence

Consider the stack $\widetilde{\mathrm{Loc}_{c}}^{\mathrm{unip}}_{G,F}$ classifying triples $(g, h, \hat{B}')$ consisting of a unipotent parameter $(g, h)$ and a Borel $\hat{B}' \subset \hat{G}$ containing $g$.

$$\widetilde{\mathrm{Loc}_{c}}^{\mathrm{unip}}_{G,F} \simeq \mathrm{Loc}_{c}{}^{\mathrm{unip}}_{G,F} \times_{\hat{G}/\hat{G}} \widetilde{\mathcal{U}}_{\hat{G}}/\hat{G}$$

# Unipotent Arithmetic Local Langlands Correspondence

Consider the stack $\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG,F}$ classifying triples $(g, h, \hat{B}')$ consisting of a unipotent parameter $(g, h)$ and a Borel $\hat{B}' \subset \hat{G}$ containing $g$.

$$\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG,F} \simeq \mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F} \times_{\hat{G}/\hat{G}} \widetilde{\mathcal{U}}_{\hat{G}}/\hat{G}$$

There is a map

$$\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG,F} \to \mathrm{St}^{\mathrm{unip}}_{\hat{G}}/\hat{G}$$

Sending a triple $(g, h, \hat{B}')$ to the triple $(g, \hat{B}', h\hat{B}')$

# Unipotent Arithmetic Local Langlands Correspondence

Consider the stack $\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG,F}$ classifying triples $(g, h, \hat{B}')$ consisting of a unipotent parameter $(g, h)$ and a Borel $\hat{B}' \subset \hat{G}$ containing $g$.

$$\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG,F} \simeq \mathrm{Loc}^{\mathrm{unip}}_{{}^cG,F} \times_{\hat{G}/\hat{G}} \widetilde{\mathcal{U}}_{\hat{G}}/\hat{G}$$

There is a map

$$\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG,F} \to \mathrm{St}^{\mathrm{unip}}_{\hat{G}}/\hat{G}$$

Sending a triple $(g, h, \hat{B}')$ to the triple $(g, \hat{B}', h\hat{B}')$

For $w \in W$ let $\mathrm{St}^{\mathrm{unip}}_{w} \subseteq \mathrm{St}^{\mathrm{unip}}_{\hat{G}}$ be the corresponding irreducible component. Denote

$$\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG,F,w} = \widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{{}^cG,F,w} \times_{\mathrm{St}^{\mathrm{unip}}_{\hat{G}}/\hat{G}} \mathrm{St}^{\mathrm{unip}}_{w}/\hat{G}$$

called the *spectral Deligne-Lusztig stack* of $w$.

# Unipotent Arithmetic Local Langlands Correspondence

Denote:

$$\pi_w^{\mathrm{unip}} \colon \widetilde{\mathrm{Loc}}{}^{\mathrm{unip}}_{{}^c G, F, w} \to \mathrm{Loc}^{\mathrm{unip}}_{{}^c G, F}$$

Then for a basic element $b \in B(G)$ we can write $w_b = \lambda_b w_{b,f}$ with $\lambda_b$ anti-dominant and $w_{b,f} \in W$.

# Unipotent Arithmetic Local Langlands Correspondence

Denote:

$$\pi_w^{\mathrm{unip}} : \widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{^cG,F,w} \to \mathrm{Loc}^{\mathrm{unip}}_{^cG,F}$$

Then for a basic element $b \in B(G)$ we can write $w_b = \lambda_b w_{b,f}$ with $\lambda_b$ anti-dominant and $w_{b,f} \in W$.

Using Bezrukavnikov's equivalence we show:

$$\mathbb{L}_G(\pi^{\mathrm{unip}}_{w_{b,f},*}(\mathcal{O}_{\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{^cG,F,w_{b,f}}}(\lambda_b)) = i_{b,*}(C_c^\infty(J_b(F)/I_b, \overline{\mathbb{Q}_\ell}))$$

Where $\mathcal{O}_{\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{^cG,F,w_{b,f}}}(\lambda_b)$ is the line bundle corresponding to $\lambda_b$ considered as a character $\hat{B} \to \mathbb{G}_m$.

# The Categorical Trace Construction

Given a $\Lambda$-linear monoidal category $\mathcal{A}$ with a monoidal endomorphism $\phi\colon \mathcal{A} \to \mathcal{A}$ we can consider the Hochschild homology

$$\mathrm{Tr}(\phi, \mathcal{A}) := \mathrm{HH}(\mathcal{A}, {}^{\phi}\mathcal{A}) = \mathcal{A} \otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}} {}^{\phi}\mathcal{A}$$

# The Categorical Trace Construction

Given a $\Lambda$-linear monoidal category $\mathcal{A}$ with a monoidal endomorphism $\phi\colon \mathcal{A} \to \mathcal{A}$ we can consider the Hochschild homology

$$\mathrm{Tr}(\phi, \mathcal{A}) := \mathrm{HH}(\mathcal{A}, {}^{\phi}\mathcal{A}) = \mathcal{A} \otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}} {}^{\phi}\mathcal{A}$$

It can be computed by realization of the (twisted) cyclic bar resolution $\mathcal{A} \simeq |\mathcal{A}^{\otimes(\bullet+2)}|$

$$\mathrm{HH}_{\bullet}(\mathcal{A}, {}^{\phi}\mathcal{A}) = \mathcal{A}^{\otimes(\bullet+2)} \otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}} {}^{\phi}\mathcal{A} \simeq \mathcal{A}^{\otimes\bullet}$$

$$d_i(a_0 \otimes \ldots, a_{i-1} \otimes a_i \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_{i-1}a_i \otimes \cdots \otimes a_n, & i \neq 0 \\ a_1 \otimes a_2 \otimes \cdots \otimes a_n\phi(a_0), & i = 0 \end{cases}$$

# The Categorical Trace of Frobenius

Now consider the situation of convolution monoidal structures. In general, these arise from:

- A map $f\colon (X, \phi_X) \to (Y, \phi_Y)$ of "geometric objects"
- $\mathcal{D}(-)$ a "sheaf theory"

# The Categorical Trace of Frobenius

Now consider the situation of convolution monoidal structures. In general, these arise from:

- A map $f\colon (X, \phi_X) \to (Y, \phi_Y)$ of "geometric objects"
- $\mathcal{D}(-)$ a "sheaf theory"

Then $\mathcal{D}(X \times_Y X)$ has a convolution monoidal structure

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, "\Delta"] \arrow[d, "m"] & (X \times_Y X) \times (X \times_Y X) \\ X \times_Y X & \end{tikzcd},$$

$$\mathcal{F} \star \mathcal{G} := m_* \Delta^! (\mathcal{F} \boxtimes \mathcal{G}), \quad \mathcal{F}, \mathcal{G} \in \mathcal{D}(X \times_Y X)$$

# The Categorical Trace of Frobenius

$$\begin{tikzcd} \mathcal{L}_\phi(Y) \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_Y"] \\ Y \arrow[r, "\mathrm{id}_Y \times \phi_Y"] & Y \end{tikzcd},$$

# The Categorical Trace of Frobenius

$$\begin{tikzcd} \mathcal{L}_\phi(Y) \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_Y"] \\ Y \arrow[r, "\mathrm{id}_Y \times \phi_Y"] & Y \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{L}_\phi(Y) \times_Y X \arrow[r] \arrow[d] & X \times_Y X \\ \mathcal{L}_\phi(Y) & \end{tikzcd}$$

Using the identification $X \times_Y \mathcal{L}_\phi(Y) \simeq X \times_{Y \times Y} Y$.

There is a natural commutative diagram:

$$\begin{tikzcd} \mathcal{D}(X \times_Y X)^{\otimes(\bullet+1)} \arrow[r] \arrow[d] & \mathrm{Tr}(\phi_*, \mathcal{D}(X \times_Y X)) \arrow[d] \\ \mathcal{D}(X^{\times_Y(\bullet+1)} \times_Y \mathcal{L}_\phi(Y)) \arrow[r] & \mathcal{D}(\mathcal{L}_\phi(Y)) \end{tikzcd}$$

# The Categorical Trace of Frobenius

$$\begin{tikzcd} \mathcal{L}_\phi(Y) \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_Y"] \\ Y \arrow[r, "\mathrm{id}_Y \times \phi_Y"] & Y \end{tikzcd}, \quad \begin{tikzcd} \mathcal{L}_\phi(Y) \times_Y X \arrow[r] \arrow[d] & X \times_Y X \\ \mathcal{L}_\phi(Y) & \end{tikzcd}$$

Using the identification $X \times_Y \mathcal{L}_\phi(Y) \simeq X \times_{Y \times Y} Y$.

There is a natural commutative diagram:

$$\begin{tikzcd} \mathcal{D}(X \times_Y X)^{\otimes(\bullet+1)} \arrow[r] \arrow[d] & \mathrm{Tr}(\phi_*, \mathcal{D}(X \times_Y X)) \arrow[d] \\ \mathcal{D}(X^{\times_Y(\bullet+1)} \times_Y \mathcal{L}_\phi(Y)) \arrow[r] & \mathcal{D}(\mathcal{L}_\phi(Y)) \end{tikzcd}$$

We can apply this paradigm to both sides of Bezrukavnikov's equivalence

# The Categorical Trace of Frobenius

Representation theory side: $(\mathbb{B}L^+\mathcal{G}, \mathrm{Fr}) \to (\mathbb{B}LG, \mathrm{Fr})$

$$\begin{tikzcd}
\mathrm{Shv}(\mathcal{I}\backslash LG/\mathcal{I}, \overline{\mathbb{Q}_\ell})^{\otimes(\bullet+1)} \arrow[r] \arrow[d] & \mathrm{Tr}(\mathrm{Fr}_*, \mathrm{Shv}(\mathcal{I}\backslash LG/\mathcal{I}, \overline{\mathbb{Q}_\ell})) \arrow[d] \\
\mathrm{Shv}(\mathrm{Hk}_\bullet(\mathrm{Sht}^{\mathrm{loc}}), \overline{\mathbb{Q}_\ell}) \arrow[r] & \mathrm{Shv}(\mathfrak{B}(G), \overline{\mathbb{Q}_\ell})
\end{tikzcd}$$

# The Categorical Trace of Frobenius

Representation theory side: $(\mathbb{B}L^+\mathcal{G}, \mathrm{Fr}) \to (\mathbb{B}LG, \mathrm{Fr})$

$$\begin{tikzcd}
\mathrm{Shv}(\mathcal{I}\backslash LG/\mathcal{I}, \overline{\mathbb{Q}_\ell})^{\otimes(\bullet+1)} \arrow[r] \arrow[d] & \mathrm{Tr}(\mathrm{Fr}_*, \mathrm{Shv}(\mathcal{I}\backslash LG/\mathcal{I}, \overline{\mathbb{Q}_\ell})) \arrow[d] \\
\mathrm{Shv}(\mathrm{Hk}_\bullet(\mathrm{Sht}^{\mathrm{loc}}), \overline{\mathbb{Q}_\ell}) \arrow[r] & \mathrm{Shv}(\mathfrak{B}(G), \overline{\mathbb{Q}_\ell})
\end{tikzcd}$$

The spectral side: $(\widetilde{\mathcal{U}}_{\hat{G}}/\hat{G}, \phi) \to (\hat{G}/\hat{G}, \phi)$

$$\begin{tikzcd}
\mathrm{Coh}(\mathrm{St}^{\mathrm{unip}}_{\hat{G}}/\hat{G})^{\otimes(\bullet+1)} \arrow[r] \arrow[d] & \mathrm{Tr}(\phi_*, \mathrm{Coh}(\mathrm{St}^{\mathrm{unip}}_{\hat{G}}/\hat{G})) \arrow[d] \\
\mathrm{Coh}\left(\left(\mathcal{U}_{\hat{G}}/\hat{G}\right)^{\times_{\hat{G}/\hat{G}}(\bullet+1)} \times_{\hat{G}/\hat{G}} \mathrm{Loc}^{\mathrm{unip}}_{c\,G,F}\right) \arrow[r] & \mathrm{Coh}(\mathrm{Loc}^{\mathrm{unip}}_{c\,G,F})
\end{tikzcd}$$

Thank you!