v-vector bundles on rigid spaces

§ Introduction

K alg closed non-arch ext of $\mathbb{Q}_p$,

X smooth rigid space over K,

$X^\diamond \longrightarrow \operatorname{Spd} K$ assoc diamond

Recall: Have hierarchy of topologies

$$\begin{tikzcd}
X_{Zar} \arrow[r, phantom, "\subseteq"] & X_{an} \arrow[r, phantom, "\subseteq"] & X_{ét} \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\parallel"] & X_{proét} \arrow[d, phantom, "\cap"] & \\
 & & X^\diamond_{ét} \arrow[r, phantom, "\subseteq"] & X^\diamond_{qproét} \arrow[r, phantom, "\subseteq"] & X^\diamond_v
\end{tikzcd}$$

in the sense of [Sch 13] w/ completed structure sheaf

All these sites carry a structure sheaf

Q: Do notions of vector bundles := loc free $\mathcal{O}_X$-modules coincide?

For τ any of the above top, let

$VB_\tau(X)$ = categ of VB on X in τ-top

$\operatorname{Pic}_\tau(X) = \{\text{line bundles on } X \text{ in } \tau\text{-top}\}/\sim$

$= H^1_\tau(X, \mathcal{O}^\times)$.

$$\rightsquigarrow \quad VB_{an}(X) \xrightarrow{(1)} VB_{ét}(X) \xrightarrow{(2)} VB_{proét}(X) \xrightarrow{(3)} VB_{qproét}(X) \xrightarrow{(4)} VB_v(X)$$

Known: - By étale descent, (1) is an isomorphism.

- $VB_{ét}(X) = VB_{ét}(X^\diamond)$ (use: $\nu_*\mathcal{O}_v = \mathcal{O}_{ét}$ for $\nu: X_v \to X_{ét}$)
- **Thm** (Kedlaya–Liu): Z perfectoid $|K$, then

  $VB_{an}(X) \cong \ldots \cong VB_v(X)$.

  Cor: (3), (4) are isomorphisms

Remaining Case: $VB_{ét}(X) \subseteq VB_{proét}(X)$.

This is $\subsetneq$ in general!

First goal: Give precise description of this discrepancy for line bundles, i.e. of $\mathrm{Pic}_{proét}(X)/\mathrm{Pic}_{ét}(X)$.

§1 Example

Reason for $\subsetneq$: $\exists$ many descent data from perf covers

Suppose we have a pro-étale perfectoid cover

$$\tilde{X} \longrightarrow X$$

that is a G-torsor for some profinite grp G. Then $\exists$ Cartan-Leray ex seq:

$$0 \longrightarrow \underset{\substack{\uparrow \\ \text{descent data on triv line bundle.}}}{H^1_{cts}(G, \mathcal{O}^\times(\tilde{X}))} \longrightarrow \mathrm{Pic}_v(X) \longrightarrow \mathrm{Pic}_{an}(\tilde{X})^G$$

Example: $X = \mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$, then $\mathrm{Pic}_{ét}(\mathbb{G}_m) = 1$.

$$\tilde{\mathbb{G}}_m = \varprojlim_{[p]} \mathbb{G}_m \longrightarrow \mathbb{G}_m$$

pro-étale $\mathbb{Z}_p(1)$-torsor. Can show: $\mathfrak{m} \subseteq \mathcal{O}_K$ max id

$$H^1_{cts}(\mathbb{Z}_p(1), \mathcal{O}^\times(\tilde{\mathbb{G}}_m)) = 1+\mathfrak{m}/\mu_{p^\infty} \xrightarrow[\sim]{\log} K$$

Upshot: $K \subseteq \mathrm{Pic}_v(\mathbb{G}_m)$

In contrast,

$$\varinjlim H^1(\mathbb{Z}/p^n(1), \mathcal{O}^\times(\tilde{\mathbb{G}}_m)) = 1.$$

§ 2 Main Theorem

Thm (H., '20): Let X be a smooth rigid space $/K$.

a) There is a natural left-ex sequence

$$0 \to \mathrm{Pic}_{an}(X) \to \mathrm{Pic}_v(X) \xrightarrow{HT\log} H^0(X, \Omega^1(-1))$$

$\nwarrow$ "Hodge-Tate logarithm"

b) The sequence is right-exact if

1) X is of pure dim 1 & paracompact

2) X is proper

c) If X is affinoid, the seq is right-exact after $[\frac{1}{p}]$ (but not in general before)

d) If $X = \mathbb{A}^n$, then $\mathrm{im}(HT\log) = H^0(\mathbb{A}^n, \Omega^1(-1))^{d=0}$.

Note: v-LB = Higgs bundles of rk 1 !

Plan: 1) Sketch of proof.

2) Applications to p-adic Simpson corresp

3) v-Picard functor

§3 Sketch of proof

The seq in a) is the Leray seq for $v: X_v \to X_{ét}$. STP:

Prop 1: $R^1 v_* \mathcal{O}^\times = \Omega^1_X(-1)$. Use:

Prop (Scholze): $R^1 v_* \mathcal{O} \xrightarrow{HT} \Omega^1_X(-1)$

relate $\mathcal{O}^\times$ to $\mathcal{O}$ via ét SES

(i) $1 \to \mu_{p^\infty} \to 1+\mathfrak{m}\mathcal{O}^+ \to \mathcal{O} \to 0$

(ii) $1 \to 1+\mathfrak{m}\mathcal{O}^+ \to \mathcal{O}^\times \longrightarrow \overline{\mathcal{O}}^\times \to 0$

$\overline{\mathcal{O}}^\times := \mathcal{O}^\times / 1+\mathfrak{m}\mathcal{O}^+$

How to think about $\overline{\mathcal{O}}^\times$?

<u>Example</u>: If X admits qc smooth formal model, special fibre $\overline{X}$:

$$H^1_{ét}(X, \overline{\mathcal{O}}^\times) = \mathrm{Pic}_{ét}(\overline{X}).$$

<u>Key calculation</u> for Prop 1:

Prop 2: $\nu: X_{proét} \to X_{ét}$

(i) $\nu^* \overline{\mathcal{O}}^\times = \overline{\mathcal{O}}^\times$ (cmp: $\nu^* \mathcal{O}^+/p = \mathcal{O}^+/p$)

(ii) $\nu_* \overline{\mathcal{O}}^\times = \overline{\mathcal{O}}^\times$, $R^1\nu_* \overline{\mathcal{O}}^\times = 1$

<u>Summary</u>: $R^1\nu_*\mathcal{O}^\times \overset{\text{Prop 2}}{=} R^1\nu_*(1+\mathfrak{m}\mathcal{O}^+) \xrightarrow[\sim]{\log} R^1\nu_*\mathcal{O} \xrightarrow[\sim]{HT} \Omega^1_X(-1)$.

$\Rightarrow$ Thm a)

Thm b) curves: $H^2_{ét}(X, \mathbb{G}_m) = 0$

Thm c) use exp SES $0 \to \mathcal{O} \to \mathcal{O}^\times[\frac{1}{p}] \to \overline{\mathcal{O}}^\times \to 1$

Thm d) explicit comput, or compare to

$$H^1_{proét}(\mathbb{A}^n, \mathbb{Q}_p) = H^0(\mathbb{A}^n, \Omega^1(-1))^{d=0}$$ (Le Bras, Colmez-Niziol)

§4 The proper case

From now on: X cndd smooth rigid space.

Idea: Create Galois cover $\tilde{X} \to X$ s.t. descent data for triv LB generate $\mathrm{Pic}_v(X)/\mathrm{Pic}_{an}(X)$

Fix $x \in X(K)$.

Def: The pro-fin-étale universal cover of X is

$$\tilde{X} = \varprojlim_{(X',x') \to (X,x)} X'$$

cnd fin ét covers w/ lift of base point

This is a spatial diamond & a profinite-étale torsor under $\pi = \pi_1(X,x)$.

Ex: - If $X=A$ is an ab var, then $\tilde{A} = \varprojlim_{[N],\, N \in \mathbb{N}} A$ is perfectoid

- If $X=C$ is a curve of genus $g \geq 1$, then $\tilde{C}$ is also perfectoid (Hansen)
- If $X = \mathbb{P}^n$, then $\tilde{X} = X$.

Prop: Let F be any one of $\mathbb{Z}/N\mathbb{Z}$ $(N \in \mathbb{Z})$, $\hat{\mathbb{Z}}$, $\mathcal{O}$, $1+\mathfrak{m}\mathcal{O}^+$, ...

$$H^0(\tilde{X}, F) = F(K) \qquad H^1_v(\tilde{X}, F) = 0.$$

(e.g. ... $\mathcal{O}) = K$.

proof: $\tilde{X}_{fét} = \varinjlim_{X' \to X} X'_{fét} = K_{fét}$. Then use Prim Comp & Log.

proof of Thm b: STP:

$$H^1_v(X, 1+\mathfrak{m}\mathcal{O}^+) \longrightarrow \mathrm{Pic}_v(X) \xrightarrow{HTlog} H^0(X, \Omega^1) \quad \text{is surjective.}$$

idea: Compare Cartan-Leray seq for $\tilde{X} \to X$ wrt $\mathrm{Log}: 1+\mathfrak{m}\mathcal{O}^+ \to \mathcal{O}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{cts}(\pi, 1+\mathfrak{m}) \arrow[r] \arrow[d, two heads, "\log"] & H^1_v(X, 1+\mathfrak{m}\mathcal{O}^+) \arrow[r] \arrow[d, "\log"] & H^1_v(\tilde{X}, 1+\mathfrak{m}\mathcal{O}^+) = 0 \arrow[d, "\overset{\mathrm{Prop}}{\cong}"] \\
0 \arrow[r] & \mathrm{Hom}_{cts}(\pi, K) \arrow[r, "\sim"] & H^1_v(X, \mathcal{O}) \arrow[r] \arrow[d, two heads, "HT"] & H^1_v(\tilde{X}, \mathcal{O}) = 0 \\
& & H^0(X, \Omega^1(-1)) &
\end{tikzcd}$$

$\pi \to K$ factors through max torfree ab pro-p-quot $= H^1_{et}(X, \mathbb{Z}_p)^\vee$ which is finite free $\mathbb{Z}_p$-mod $\to$ can lift generators. □

More generally:

Observation: The CL-spectral seq for GL_n gives

$$0 \to \underset{\left\{\text{cts reps } \pi \to GL_n(K)\right\}/\text{conj}}{\overset{}{\underset{\parallel}{H^1_{cts}(\pi, GL_n(K))}}} \to H^1_v(X, GL_n(\mathcal{O})) \to H^1_v(\tilde{X}, GL_n(\mathcal{O}))^\pi$$

Def: A v-VB on X is called "proét" if it is trivialised by some pro-fin-ét cover; equiv by $\tilde{X} \to X$

⇝ get category equivalence

$$\mathrm{Rep}_\pi(K) := \left\{\begin{matrix}\text{cts reps of } \pi \\ \text{on f.d. } K\text{-vect sp}\end{matrix}\right\} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{proét} \\ v\text{-VB}\end{matrix}\right\}$$
$$V(\tilde{X}) \longleftarrow\!\shortmid \; V$$

Q: Can one characterise when a rep $\rho \leftrightarrow$ analytic VB?

This generalises a constr in:

§5 The p-adic Simpson correspondence

Def: A Higgs bundle on X is a pair (E, θ) where

- E an analytic VB on X
- $\theta \in H^0(X, \mathrm{End}(E) \otimes \Omega^1_X(-1))$ s.t. $\theta \wedge \theta = 0$

Conj: There is a categ equivalence

$$\mathrm{Rep}_\pi(K) \xleftrightarrow{\sim} \; ? \subseteq \{\text{Higgs bundles on } X\}$$

Deninger-Werner, Würthen, Mann-Werner : $X / \mathbb{C}_p$

$$\left\{\begin{array}{l}\text{analytic VB} \\ \text{w/ numerically} \\ \text{flat reduction}\end{array}\right\} \longleftrightarrow \mathrm{Rep}_\pi(K)$$

This is case of $\theta = 0$

Faltings, Abbes-Gros-Tsuji, Liu-Zhu, Wang (21), ...:

$$\begin{tikzcd}
\mathrm{Rep}_\pi(K) \arrow[r, leftrightarrow] & \textcircled{?} \\
\rotatebox{-90}{$\subseteq$} & \rotatebox{-90}{$\subseteq$} \\
\{\text{"generalised reps"}\} \arrow[r, dashed, no head] & \{\text{Higgs bundles}\} \\
\cong \{\text{proét-loc-free } \mathcal{O}_X^+\text{-mods}\}/_{\text{isog}} & \rotatebox{90}{$\subseteq$} \\
\rotatebox{90}{$\subseteq$} & \\
\{\text{"small" gen reps}\} \arrow[r, leftrightarrow, "\sim"] & \{\text{"small" Higgs bundles}\}
\end{tikzcd}$$

Our Main Thm gives case of rk = 1:

<u>Cor</u> (p-adic Simpson corresp of rk 1):

∃ functorial SES

$$0 \longrightarrow \mathrm{Pic}^{\mathrm{proét}}_{\mathrm{an}}(X) \longrightarrow \mathrm{Hom}_{\mathrm{cts}}(\pi, K^\times) \xrightarrow{\mathrm{HTlog}} H^0(X, \Omega^1(-1))$$

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{cts}}(\pi, K^\times) \arrow[d, equal] \\ \mathrm{Pic}^{\mathrm{proét}}_{v}(X) \end{tikzcd}$$

any choice of splitting of log & HT induces

$$\mathrm{Rep}^{1\text{-dim}}_{\pi} K \xleftrightarrow{\ \sim\ } \{\text{proét Higgs bundles of rk } 1\}$$

⇝ in rk 1, $\textcircled{?} = \underline{\underline{\text{proét}}}$ Higgs

Q: Can $\mathrm{Pic}^{\mathrm{proét}}_{\mathrm{an}}(X)$ be made more explicit? ⟶ Yes!

<u>Thm</u> 1) Assume that the Picard functor of X is represented by a rigid grp var $\underline{\mathrm{Pic}}$. Then

$$\mathrm{Pic}^{\mathrm{proét}}_{\mathrm{an}}(X) = \underline{\mathrm{Pic}}(K)^{tt} := \{x \mid x^{n!} \to 0 \text{ for } n \to \infty\}$$

2) If $K = \mathbb{C}_p$, and $\underline{\mathrm{Pic}}^0$ is abeloid (e.g. if X algebraic), this is $=$ "$\mathrm{Pic}^{\tau}_{\mathrm{an}}(X) = \mathrm{Pic}^0 + NS_{\mathrm{tor}}$"

In general, $\underline{\mathrm{Pic}}^{tt} := \mathcal{H}om(\underline{\hat{\mathbb{Z}}}, \underline{\mathrm{Pic}})$ is an open subgrp, closely related to Fargues analytic p-div grps.

3) If $A = \underline{\mathrm{Pic}}^0$ is abeloid, then induced map

$$A(K)^{tt} \longrightarrow \mathrm{Hom}(\pi, K^\times)$$

is the analytic cont of the Weil pairing.

Deduce Thm from:

"Thm (H, Feb 21)": Let $f: X \to \mathrm{Spa}\, K$ be cndd smooth proper. Assume $\underline{\mathrm{Pic}}_{\acute{e}t}$ is rep by a rigid grp var. Then

1) The diamantine Picard functor
$$\underline{\mathrm{Pic}}^{\diamond}_{\acute{e}t} := R^1 f^{\diamond}_{\acute{e}t *} \mathbb{G}_m : \left(\text{perfectoid } Y \longmapsto \mathrm{Pic}_{\acute{e}t}(X \times Y)\right)^{sh}$$
is represented by $\left(\underline{\mathrm{Pic}}_{\acute{e}t}\right)^{\diamond}$

2) The v-Picard functor
$$\underline{\mathrm{Pic}}^{\diamond}_{v} := R^1 f^{\diamond}_{v *} \mathbb{G}_m : \quad \left(\text{perfectoid } Y \longmapsto \mathrm{Pic}_v(X \times Y)\right)^{sh}$$
sits in a SES of rigid grp vars
$$0 \to \underline{\mathrm{Pic}}^{\diamond}_{\acute{e}t} \longrightarrow \underline{\mathrm{Pic}}^{\diamond}_{v} \xrightarrow{HT\log} H^0(X, \Omega^1(-1)) \otimes \mathbb{G}_a \longrightarrow 0$$
$\rightsquigarrow$ geom version of Main Thm.