A p-adic transcendence criterion for CM Galois representations

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1. A simple p-adic transcendence result

Theorem (Schneider) If $\tau \in \mathbb{H} \cap \overline{\mathbb{Q}}$ and $j(\tau) \in \overline{\mathbb{Q}}$ then τ is imaginary quadratic.

Example $j(3^{1/3} i)$ is transcendental / $\mathbb{Q}$

Theorem (H., 2018): If $K/\breve{\mathbb{Q}}_p$ is a finite extension and $\mathcal{G}/\mathcal{O}_K$ is a 1-dim'l p-divisible group whose Hodge-Tate filtration is algebraic over $\breve{\mathbb{Q}}_p$ then $\mathcal{G}_{\mathcal{O}_{\mathbb{C}_p}}$ has CM.

Restatement of Schneider. If $K/\mathbb{Q}$ is a finite extension and E/K is an elliptic curve s.t. Hodge filtration on $H_1(E(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C}$ is defined over $\overline{\mathbb{Q}}$, then $E_{\mathbb{C}}$ has CM.

Notation $\breve{\mathbb{Q}}_p = W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$ $\quad \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p} \supseteq \breve{\mathbb{Q}}_p$

$$T_p\mathcal{G} = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mathcal{G}_{\mathcal{O}_{\mathbb{C}_p}})$$

$$= \mathrm{Hom}(\mathcal{G}^\vee_{\mathcal{O}_{\mathbb{C}_p}}, \mathbb{Q}_p/\mathbb{Z}_p)$$

$$T_p\mathcal{G} \to \omega_{\mathcal{G}^\vee_{\mathcal{O}_{\mathbb{C}_p}}}$$

$$x \longrightarrow x^* \frac{dt}{t}$$

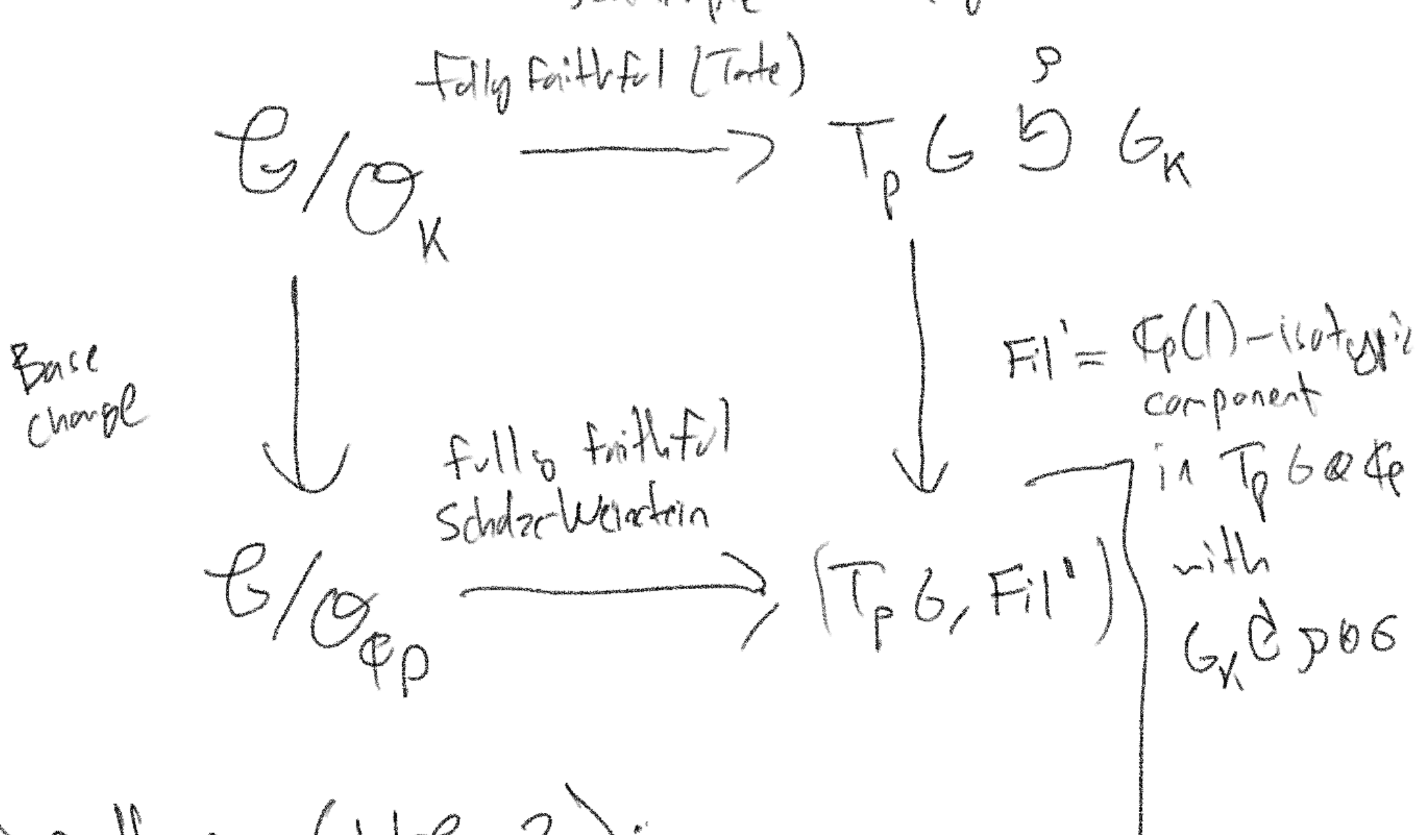
$$T_p\mathcal{G} \otimes \mathbb{C}_p \to \omega_{\mathcal{G}^\vee} \otimes \mathbb{C}_p$$

~~Hodge Filt~~ Fil' is Kernel of this map.

Fil' is algebraic if it comes from

$$T_p\mathcal{G} \otimes \overline{\mathbb{Q}_p}$$

$\mathcal{G}_{\mathcal{O}_{\mathbb{C}_p}}$ has CM if $\mathrm{End}(\mathcal{G}_{\mathcal{O}_{\mathbb{C}_p}})[\frac{1}{p}]$ contains a commutative semisimple $\mathbb{Q}$-algebra of $\dim = \mathrm{ht}\,\mathcal{G}$.



Proof of theorem ($\mathrm{HT}\Theta = \mathcal{L}$).

$\mathcal{G}_{\overline{\mathbb{F}}_p}$ = ordinary $\mathcal{N}_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$ | supersingular (isoclinic of slope $1/2$)

$\mathcal{N}_{p^\infty} \to \mathcal{G} \to \mathbb{Q}_p/\mathbb{Z}_p$

$\mathbb{Q}_p(1) \to T_p\mathcal{G}[\frac{1}{p}] \to \mathfrak{g}_p$

HT Filtration - deformed $\mathbb{Q}_p$.

$\mathfrak{g}_p \to \mathfrak{g}_p^2 \to \mathfrak{g}_0$

can split however

$\Rightarrow$

$\mathcal{G}_{\mathcal{O}_0} = \mathcal{N}_{p^\infty} \times \mathbb{Q}_1/\mathbb{Z}_p$

Replace K with a finite extension so that HT filtration is K-rational, i.e.

$\mathrm{Fil}^1 \subseteq T_p\mathcal{G} \otimes \mathfrak{g}_p$ — Action of G_K.

$1 \otimes \sigma$ preserves Fil^1

$p \otimes \sigma$ preserves Fil^1

$2/3$ argument $\Rightarrow$

$p \otimes 1$ preserves Fil^1

↑ linear action of G_K.

Fil^1 is a 1-dimensional subspace of $T_p\mathcal{G} \otimes \mathfrak{g}_p$ preserved by $p \otimes 1$ and it's not $\mathfrak{g}_p$-rational (we'd be in the ordinary)

Take some non-trivial conjugate $\mathrm{Fil}^1 \oplus (1 \otimes \tau)(\mathrm{Fil}^1)$ $\tau \in G_{\mathbb{Q}_p}$

preserved by $p \otimes 1$

... finite étale ...

so $\mathrm{Im}(\gamma)$ semisimple & abelian

(Tate) $\Rightarrow \mathrm{End}(G_{\bar{k}}) \otimes \mathbb{Q} = \mathrm{End}_{G_k}(T_p G) \otimes \mathbb{Q}$

↑ contains a maximal rank commutative sub-algebra

Different version of Theorem: — also true

$\mathcal{M}_{LT,n}$ — Lubin-Tate moduli space for height n 1-dim'l formal group.

$\xrightarrow{\pi_{H/I}} \mathbb{P}^{n-1}$

$\downarrow \pi_{GM}(\pi_{GH})$

$\mathbb{P}^{n-1}$

If $x \in \mathcal{M}_{LT,n}(\bar{\mathbb{F}}_p)$

s.t. $\pi_{H/I}(x) \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$

$\pi_{GM}(x) \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$

then x is a special point

(parameterizes CM formal group $\leftrightarrow$)

$T \to GL_n$

$(\curvearrowright D_{\mathbb{Q}_p}^{\times})$

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2. Generalizations in arch. setting / why?

1. Tretkoff (Cohen, Shiga-Wolfart)

Generalizes Schneider to abelian varieties / Hodge type Shimura varieties

$$\begin{array}{ccc} D & \xrightarrow{i} & Fl \\ \pi \downarrow & & \\ Sh(\Phi) & & \end{array}$$

If $x \in D$

$i(x) \in Fl(\overline{\mathbb{Q}})$

$\pi(x) \in Sh(\overline{\mathbb{Q}})$

$\rightsquigarrow$ x is a special point

2. ~~Pink~~ Bi-algebraicity of higher dim'l subvarieties

Theorem (Ullmo & Yafaev): ← Abelian type setting

If $Z \subseteq Fl_{\overline{\mathbb{Q}}}$

and $\pi(i^{-1}(Z))$ is

a Zariski closed subset

of $Sh_{\overline{\mathbb{Q}}}$

Then $\pi(i^{-1}(Z))$ is a special subvariety.

Zilber-Pink for mixed Shimura varieties

(Weak version of André-Linberman-Weierstrass type theorem — classifies maximal algebraic subsets of $\pi^{-1}(Y)$ $Y \subseteq Sh(\Phi)$ Zariski closed)

Key ingredient in the Pila-Zamir strategy for the André-Oort

(Cutler bialgebraic conjectures)
e.g. Main-Manfredo
Zilber-Pink
// unlikely intersections

bialgebraic

3. p-adic generalizations

Theorem (H.) If $K/\mathbb{Q}_p$ fin. ext'n
and $\mathcal{G}/\mathcal{O}_K$ is a p-divisible group
then $\mathcal{X}_{\mathcal{G}}$ has algebraic HT filtration
$\Rightarrow$ $\mathcal{G}|_{\mathcal{O}_{\mathbb{C}_p}}$ has CM.

Remark Isocrystal $\rightarrow$ General case
much trickier because
slope filtrations don't lift.

Generalize $\rightsquigarrow$ Replace ~~p~~-div. group
w/ a crystalline representation

$$\begin{array}{ccccc} \mathcal{G}_{\mathcal{O}_K} & \longrightarrow & T_p\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_K & \longhookrightarrow & (V, \rho) \\ & & & & \text{crystalline} \\ \downarrow & & \downarrow & & \downarrow \text{base change} \\ & & & & \text{to } \mathbb{C}_p \end{array}$$

$\mathcal{L}/\mathcal{O}_{\mathcal{L}}$ $\longrightarrow$ $(T_p G, \mathrm{Fil}^1)$ $\xhookrightarrow{\text{Minuscule cocharacter}}$ $(V, \mathcal{L}_{dR})$

Fil $\mathbb{Q}_p$-vector spaces equipped with a B_{dR}^+ lattice $\mathcal{L}_{dR}$ in $V \otimes B_{dR}$

(trace on special fiber $V \otimes \mathcal{L}_1 = V \otimes B_{dR}^+/t$)

(V, ρ) $\rightsquigarrow$ $\left(V, (V \otimes B_{dR})^{G_K} \otimes B_{dR}^+\right)$

$\left(V, D_{dR}(V) \otimes B_{dR}^+\right)$

$\| \wr$

$D_{dR}(V) \otimes B_{dR}$

$\|$

$\wr V \otimes B_{dR}$

Theorem If (V, ρ) is a potentially crystalline representation of G_K $K/\mathbb{Q}_p$ with algebraic de Rham lattice ($\exists$ $\mathcal{L}_{dR} \subseteq V \otimes B_{dR}$ preserved by $G_{K'}$ K'/K fin. ext. ($\subseteq \bar{K}$))

($\subseteq$ HT filt algebraic for p-divisible group)

then (V, ρ_{dR}) w/ $\mathrm{Fil}^{\cdot}$.

(V, ρ)

Theorem

$$\begin{array}{ccc} \mathcal{M}_{(G,X,\mu)} & \xrightarrow{\Pi_{HT}} & Gr_{X,\mu} \\ \downarrow \Pi_{dR} & & \\ Gr_X & & \end{array}$$

If b is basic,

$\Pi_{HT}(x) \subseteq Gr_{X,\mu^{-1}}(\overline{\mathbb{Q}}_p)$

and

$\Pi_{dR}(x) \in Gr_X(\overline{\mathbb{Q}}_p)$ then x is a special point.

Also a version where b is not basic replace special with unipotent group orbit of special.

Theorem If (V, ρ) is a crystalline

p-adic field K ⇝

with K-rational de Rham lattice

then

(1) The slope Filtration "lifts" on $D_{dR}(V)$ (induced by filtration $(V, \mathcal{F})$).

(2) The isoclinic quotients are abelian after restriction to inertia.

Remark Slope filtration lifting ($\rightsquigarrow$) weakly admissible $\neq$ admissible in general prec. to $\mathbb{Q}_p$

J_{Borel}

Proof: 2/3 trick again (1), (2)

$\dashv$ embd Galois repn $\longrightarrow$ G_K-eq. vector bundles on the Fargues-Fontaine curve.

$(V, \mathcal{F})$ $\rightsquigarrow$ $X = FF$ curve

$\infty = \operatorname{Spec} \mathbb{C}_p$

$\hat{X}_\infty = B_{dR}^+$

$G_K \curvearrowright X$

$\leftsquigarrow$ $Y = \operatorname{Spec}(W(\mathcal{O}_{\mathbb{C}_p}), V(p[\pi]))$

$\in$

$H^0(X, \mathcal{O}) = \mathbb{Q}_p$

$\mathrm{Spa}(R_{inf}, R_{inf}^+)$

$- V(p^{\flat})$

G_K $V \otimes \mathcal{O}$

$p \otimes \sigma$

de Rham

$V \otimes \mathcal{O}|_{\hat{X}_\infty - \infty}$

has a G_K-

eq. trivialization

$\mathcal{L}_{dR} =$

$= \mathrm{Spec}\, B_{dR}$

Vector bundle

Glue $V \otimes \mathcal{O}|_{X - \infty}$ to $\mathcal{L}_{dR}$ over $\hat{X}_\infty$ over $\hat{X}_\infty$

over $\hat{X}_\infty - \infty$

$G_K \curvearrowright$ $G_K \curvearrowright$

$V \dashrightarrow W$ in crystalline setting

$W \cong D_{cris}(V) \otimes \mathcal{O}_Y$

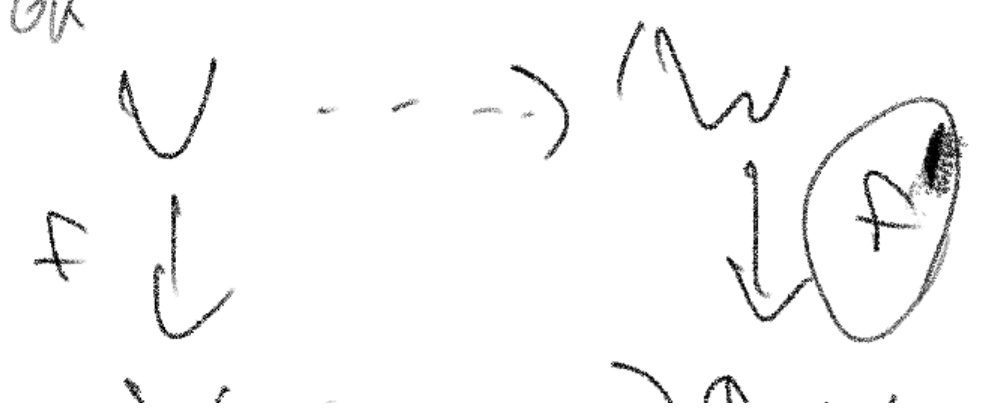
e^z descent

with G_K-action

$\otimes$

If $f \in \mathrm{End}(V)$ $(f \in \mathrm{End}(V, \mathcal{L}_{dR}))$

f preserves $\mathcal{L}_{dR}$



$f' \in \mathrm{End}(W)$

CM-Hodge structure