p-adic Deligne-Lusztig spaces

Intro:

⊛ G red. gp. / $\mathbb{F}_q$, $\underbrace{T \subseteq B}_{\mathbb{F}_q\text{-rat'l}} \subseteq G$, $W = N_G(T)/T$ Weyl group

($\Leftrightarrow G/\overline{\mathbb{F}}_q$ + Frob. σ)

$$G \curvearrowright (G/B)^2 = \coprod_{w\in W} \mathcal{O}(w) \quad \text{Bruhat decomposition.}$$

$\llcorner \{g \xrightarrow{w} h\}$

$\llcorner \Leftrightarrow g^{-1}h \in BwB$.

⊛ DL-variety for G, $w \in W$:

$$\begin{tikzcd} G^\sigma = G(\mathbb{F}_q) \arrow[r, phantom, "\curvearrowright"] & X_w \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}(w) \arrow[d] \\ & G/B \arrow[r, hook, "(id,\sigma)"'] & (G/B)^2 \end{tikzcd}$$

⊙ for $\dot{w} \mapsto w$, similar: $\dot{X}_{\dot{w}} \subseteq G/U$, U = unip. rad. of B.

$$G \curvearrowright \dot{X}_{\dot{w}} \xrightarrow[T_w(\mathbb{F}_q)]{} X_w$$

, where T_w = $\mathbb{F}_q$-form of T , with Frob. $\mathrm{Ad}(w)\circ\sigma$.

$G(\mathbb{F}_q)\times T_w(\mathbb{F}_q)$

⊙ D.-L. '76: ℓ-adic Cohomology of all $\dot{X}_{\dot{w}}$ ($\forall w$) realizes all irrep. of $G(\mathbb{F}_q)$

⊙ **Fact**: X_w (& $\dot{X}_{\dot{w}}$) are smooth $\overline{\mathbb{F}}_q$-varieties of dim $\ell(w)$.

⊙ Thm. (Orlik-Rapoport ($\simeq$2008); He; Bonnafé-Rouquier ($\simeq$2008)).

If w minimal in its σ-conj. class in W, then X_w affine.
(& $\dot{X}_{\dot{w}}$)

④ This talk: generalize this to $G/\mathbb{Q}_p$.

Problem: $X_0/\mathbb{Q}_p \to X = X_0 \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$: no Frobenius on X.

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① Loop functor.

⊙ p prime, work over $\mathbb{Q}_p$ (may replace $\mathbb{Q}_p$ by any non-arch local field)

fix $\breve{\mathbb{Q}}_p := \underbrace{\widehat{\mathbb{Q}_p^{nr}} / \mathbb{Q}_p}_{\sigma \text{ Frobenius}}$ & corresp. $L = \overline{\mathbb{F}}_p/\mathbb{F}_p$.

Perf := perfect $\mathbb{F}_p$-alg. ; $k \in$ Perf $\rightsquigarrow$ Perf$_k$

$R \in$ Perf $\rightsquigarrow$ $\underline{\underline{W(R)}}$ = p-typical Witt ring : top. $\mathbb{Z}_p$-Alg, p-ad. cpl
& separated.

⊙ $X/\mathbb{Q}_p$ scheme, the loop space of X:

$$LX : \mathrm{Perf} \longrightarrow \mathrm{Sets},\ R \mapsto X(W(R)[\tfrac{1}{p}])$$

e.g. $LX(k) = X(\breve{\mathbb{Q}}_p)$.

Fact: X affine of fin. type $/\mathbb{Q}_p$, then LX ind-scheme.

Expl. $X = \mathbb{A}^1_{\mathbb{Q}_p}$, $R \in \mathrm{Perf}$.

Teichmüller lift;

$$LX(R) = W(R)[\tfrac{1}{p}] = \Big\{ \sum_{n \gg -\infty} [x_n]\, p^n \;\Big|\; x_n \in R \Big\}.$$

$$\Rightarrow LX \cong \varinjlim_{N \to \infty} \prod_{n=-N}^{\infty} \mathbb{A}^{1,\mathrm{perf}}_{\mathbb{F}_p}$$

$L^+X : R \mapsto X(W(R))$

Expl: $X = \mathbb{P}^1$: $L\mathbb{P}^1(k) = \mathbb{P}^1(\breve{\mathbb{Q}}_p) = \mathbb{P}^1(\breve{\mathbb{Z}}_p) = L^+\mathbb{P}^1(k)$ ($\forall$ field$/k$)

$L^+\mathbb{P}^1$ repres: $\varprojlim_i L^+_i \mathbb{P}^1$,

$$\begin{tikzcd} \cdots \arrow[r, dashed] & L^+_2\mathbb{P}^1 \arrow[r, "\mathbb{A}^1\text{-bdl}"'] & L^+_1\mathbb{P}^1 \arrow[d, equal] \\ & & \mathbb{P}^1 \end{tikzcd}$$

but $R = k[[t^{1/p^\infty}]] \in \mathrm{Perf}$, $L\mathbb{P}^1(R) \neq L^+\mathbb{P}^1(R)$.

Frobenius: if $X_0/\mathbb{Q}_p$, $X = X \times_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p$, then

$LX = LX_0 \times_{\mathbb{F}_p} k$ (as $L(\cdot)$ comm. with fiber prod. & $L(pt) = pt$)

carries a geom. Frob. $\sigma : LX \to LX$. (in expl. above $x_n \mapsto x_n^p$)

④ p-adic DL-spaces

$G/\mathbb{Q}_p$ unramified red. ($\triangleq G/\breve{\mathbb{Q}}_p$ + Frob. σ on $G(\breve{\mathbb{Q}}_p)$)

$T \subseteq B \subseteq G$; W ; $(G/B)^2 = \coprod_{w \in W} \mathcal{O}(w)$.

$\mathbb{Q}_p$-rat'l

<u>rel. position</u>: $S \in \mathrm{Perf}_k$, $g, h \in L(G/B)(S)$

$$\begin{tikzcd} g \arrow[r, "w"] & h \end{tikzcd} \;:\Longleftrightarrow\; \begin{tikzcd} S \arrow[r, "(g,h)"] \arrow[dr, "\exists"'] & L(G/B)^2 \\ & L\mathcal{O}(w) \arrow[u] \end{tikzcd} \;\Longleftrightarrow\; \begin{tikzcd} W(S)[\frac{1}{p}] \arrow[r, "(g,h)"] \arrow[dr] & (G/B)^2 \\ & \mathcal{O}(w) \arrow[u] \end{tikzcd}$$

<u>Def.</u> (Scholze) $b \in G(\breve{\mathbb{Q}}_p)$, $w \in W$

$$\begin{tikzcd} X_w(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L\mathcal{O}(w) \arrow[d] \\ L(G/B) \arrow[r, "{(\mathrm{id}, b\sigma)}"'] & L(G/B)^2 \end{tikzcd}$$

as functors on Perf.

<u>similarly</u>: $\dot{X}_{\dot{w}}(b)$ $\hookrightarrow L(G/U)$.

$\dot{w} \mapsto w$

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⑤ Arc-sheaves

<u>arc-topology</u> (Bhatt–Mathew): $S' \to S$ (qcqs schemes) is arc-cover if any immediate specialization in S lifts to S'.

Thm 1. (I.'20) X qu-proj. / $\breve{\mathbb{Q}}_p$. Then LX arc-sheaf on Perf_k.

Cor. $X_w(b)$ ($\&\ \dot{X}_{\dot{w}}(b)$) arc-sheaves.

Thm 2. (I.'20) $LG \twoheadrightarrow L(G/B)$ surjection for v-top.

Rem.
- Bouthier – Česnavičius: true for ét top but only in mixed char;
- Anschütz: ∀ parabolic B.
- use:

$$\begin{tikzcd} LG \arrow[r, two heads] \arrow[dr] & L(G/U) \arrow[d] \\ & L(G/B) \end{tikzcd} \implies \begin{tikzcd} \coprod \dot{X}_{\dot{w}}(b) \arrow[d, "\dot{w}\mapsto w"'] \\ X_w(b) \end{tikzcd}$$

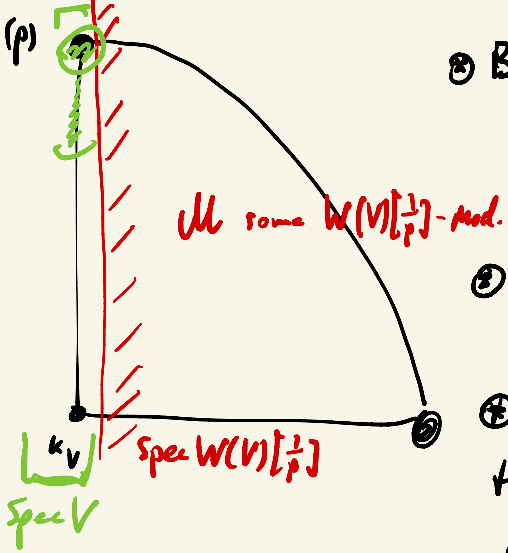
⊛ Sketch of Pf of Thm 2: suffices $H^1_{ét}(\mathrm{Spec}\, W(A)[\frac{1}{p}], \overset{T}{\cancel{B}})=0$

for $A \in \mathrm{Perf}$, small enough in v-top.; i.e. suffices:

⊙ **Thm.** (I.) Let $A \in \mathrm{Perf}_k$ be s.t. each comp. of $\mathrm{Spec}\, A$ is the spectrum of a val. ring. Then ∀ fin loc. free $W(A)[\frac{1}{p}]$-Module is free. In part., $H^1_{ét}(\mathrm{Spec}\, W(A)[\frac{1}{p}], GL_n)=0$.

Proof sketch:

⊙ This reduces by Noeth. approx. + Gabber-Ramero to $A=V$ a val. ring. Then:



⊛ By Beauville-Laszlo: extend to the point (p).

⊛ Then by Noeth. approx. to nbhd. of (p)

⊛ Then by a result of Kedlaya to all of Spec $W(V)$. Now, a fin.-proj. $W(V)$-module is free.

⊛ Sketch of proof of Thm 1. two steps.

Step 1: Descent for vector bundles.

Prop. The fibered cat. $R \in \mathrm{Perf} \mapsto \left(\begin{array}{l}\text{loc. free fin.}\\ W(R)[\frac{1}{p}]\text{-Mod.}\end{array}\right)$ is an arc-stack.

Cor. $LP^n_{\mathbb{Q}_p}$ is an arc-sheaf.

sketch: Perf'd := cat. of all perfectoid spaces.

Def. $\{f_i : X_i \to Y\}_{i \in I}$ in Perf'd arc-cover if ∀ qc open $V \subseteq Y$ $\exists I_V \subseteq I$ fin. + a qc. $U_i \subseteq X_i$ $(\forall i \in I_V)$ st.

∀ rk 1 pt. of V comes from a rk 1 pt. of some U_i.

Idea: v-descent ∀ object only depending on $\mathcal{O}_X$ (not on $\mathcal{O}_X^+$) should generalize to arc-descent. E.g.:

Lemma. $X \in \mathrm{Perf'd} \mapsto \{\text{loc. free + fin. } \mathcal{O}_X\text{-Mod.}\}$ is an arc-stack.

Proof. Scholze-Weinstein (same claim for v-top) + ε. □

(+) Now, let $R \to R'$ arc cover in Perf.:

$$\begin{tikzcd}
\mathrm{Spec}\,R' \arrow[d, "\text{arc-cover}"] \arrow[r, phantom, "\overset{\text{key pt}}{\overset{!}{\Longrightarrow}}"] & \text{arc-cover} & U' = \mathrm{Spa}(W(R')[\frac{1}{p}], W(R')) \arrow[d, "\text{arc-cover}"'] \arrow[r, Rightarrow] & \tilde{U}' \arrow[l] \arrow[d, "\text{arc cover}"] \arrow[dl, phantom, "\llcorner"] \\
\mathrm{Spec}\,R & & U = \mathrm{Spa}(W(R)[\frac{1}{p}], W(R)) & \tilde{U} \arrow[l, "\text{pass to some perf'd cover}"]
\end{tikzcd}$$

$\tilde{U}$ = aff'd perf'd cover of U, e.g.

$$\tilde{U} = U \times_{\mathrm{Spa}\,\mathbb{Z}_p} \mathrm{Spa}\,\mathbb{Z}_p[p^{1/p^\infty}]^\wedge_p .$$

(*) Using ($\tilde{U}' \to \tilde{U}$ arc-cover) + (prev. LM.) + (an argument from Berkeley notes (for v-top. there)) we can descend a VB over U' to U.

(‡) (**Key**) **Prop.** $R \to R'$ arc-cover $\in$ Perf. Then

$\mathrm{Spa}_{\le 1} W(R') \longrightarrow \mathrm{Spa}_{\le 1} W(R)$ surjective.

Proof. technical; red. to case $R \to R'$ f.flat map of val rings. □

(Prop. above proven, step 1 done.)

Step 2.

Cor (to Key Prop.) $R \to R'$ arc-cover $\in \mathrm{Perf}$. Then the image of $W' = \mathrm{Spec}\, W(R')[\frac{1}{p}] \to \mathrm{Spec}\, W(R)[\frac{1}{p}] = W$ contains all closed points.

Proof idea: realize a max. id. as supp. of a val. of rk 1, then lift. □

Cor. 2: $Y \to X$ immersion of $\mathbb{Q}_p$-schemes. If LX arc-sheaf, then LY too.

Pf: easy. cl. imm.: use $W' \to W$ dominant;

op. imm.: use last cor. □

□ (Thm 1.)

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⑤ Properties of $X_w(b)$.

1) Action T loc. profin. set (+ 2nd-countable)

$\rightsquigarrow \underline{T} : \mathrm{Perf}_k \longrightarrow \mathrm{Sets}, \; S \mapsto C^0(|S|, T)$

If T profin.: $\underline{T} \cong \mathrm{Spec}\, C^0(T, k)$

Lemma. T loc. profin., X qu-proj. $/\breve{\mathbb{Q}}_p$. Then

$$\mathrm{Hom}_{Shv(Perf_k)}(\underline{T}, LX) = C^0(T, \underbrace{X(\breve{\mathbb{Q}}_p)}_{p\text{-adic top.}}).$$

Proof. red. to T profin.; then:

$$\mathrm{Hom}_{Shv}(\underline{T}, LX) = LX(C^0(T,k))$$ as $\underline{T} = \mathrm{Spec}\, C^0(T,k)$

$$= X(W(C^0(T,k))[\tfrac{1}{p}])$$ by def of LX.

$$= X(C^0_{p\text{-adic}}(T, \breve{\mathbb{Q}}_p))$$ (as $W(C^0(T,k)) = C^0_{p\text{-ad}}(\sim)$ using univ. property of $W(-)$. or expl. comp.

$$= \mathrm{Cont}(T, X(\breve{\mathbb{Q}}_p)). \quad \square$$

⊛ $b \in G(\breve{\mathbb{Q}}_p)$:

$$R/\mathbb{Q}_p \mapsto G_b(R) := \{ g \in G(R \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p) \mid g b\sigma = b\sigma g \}$$

$G_b/\mathbb{Q}_p$ inner form of a Levi of G.

$$\begin{tikzcd} \underline{G_b(\mathbb{Q}_p)} \arrow[r, hook] \arrow[dr, "\cong"'] & LG \\ & {} \arrow[u] \end{tikzcd}$$

(ind. by $G_b(\mathbb{Q}_p) \subseteq G(\breve{\mathbb{Q}}_p)$ & prev. Lemma.)

$$\overset{\searrow\ \ V}{LG^{b\sigma}} := \mathrm{Eq.}\left(LG \overset{id}{\underset{\mathrm{Int}(b)\circ\sigma}{\rightrightarrows}} LG\right)$$

careful, as LG not qc.

<u>Fact</u>. LG-action on $L(G/B)$ restricts to an $\underline{G_b(\mathbb{Q}_p)}$-action on $X_w(b)$. (<u>proof</u>: purely formal, on R-pts.)

2) $\coprod$-decomposition

$\underset{\substack{\circlearrowleft\\ \sigma}}{S} \subseteq W$ set of simple reflections ($\hat{=}$ choice of B) (as $\sigma(B) = B$)

<u>Lemma</u>. b, w, $I \subseteq S$ σ-stable containing $\mathrm{supp}(w)$. Then

$$\begin{tikzcd} X_w(b) \arrow[r] \arrow[rrd] & L(G/B) \arrow[r] & L(G/P_I) \\ & & L(G/P_I)^{b\sigma} \end{tikzcd}$$

factors through $L(G/P_I)^{b\sigma} := \mathrm{Eq.}(id, b\sigma : \underset{\circlearrowleft}{L(G/P_I)})$.

(formal)

<u>Prop</u>. 1) $\forall f \in \mathrm{Perf}$ alg. cl. field.

$$(G/P_I)(W(f)[\tfrac{1}{p}])^{b\sigma} = \coprod_{i=1}^{r} G_{b_i}(\mathbb{Q}_p)/P_{I,b_i}(\mathbb{Q}_p),$$

where $\underset{\substack{\shortparallel \\ \{g^{-1}b\sigma(g) \mid g \in G(\breve{\mathbb{Q}}_p)\}}}{[b]_G} \cap P_I(\breve{\mathbb{Q}}_p) = \coprod_{i=1}^{r} [b_i]_{P_I}$ (fin. many: Kottwitz)

In part., $(G/P_I)(\breve{\mathbb{Q}}_p)^{b\sigma}$ profinite

2) Nat'l map $\underline{(G/P_I)(\breve{\mathbb{Q}}_p)}^{bo} \longrightarrow L(G/P_I)^{bo}$ is an isomorphism.

(by Lemma above)

Proof. 1): (statement on geom. pts) Gal. coh. argument.

2): Use

⊛ **Lemma.** $f: \underset{qcqs}{\mathcal{F}} \longrightarrow \underset{qs}{\mathcal{G}}$ map of v-sheaves. TFAE:

(a) f isom.

(b) $\forall\, V \in \mathrm{Perf}$ val ring (w. alg. cl res. field),
$f(V): \mathcal{F}(V) \longrightarrow \mathcal{G}(V)$ bijective.

⊛ Verify Cond. (B) of Lemma:

$U = \operatorname{Spec} V \ni \eta =$ gen. pt. Then

$$\begin{tikzcd}
\underline{(G/P_I)(\breve{\mathbb{Q}}_p)}^{bo}(U) \arrow[r, equal, "\text{as } I \text{ loc. const.}"] \arrow[d, equal] & \underline{(G/P_I)(\mathbb{Q}_p)}^{bo}(\eta) \arrow[d, equal, "\text{by 1)}"] \\
L(G/P_I)^{bo}(U) \arrow[r, hook] & L(G/P_I)^{bo}(\eta)
\end{tikzcd}$$

use X sep. $\Rightarrow LX$ sep.

must be $=$, as all others are.

□

$\Rightarrow$ **Thm.** In above sit. have:

$$X_w^G(b) \cong \coprod_{i=1}^{r} \underline{G_{b_i}(\mathbb{Q}_p)/P_{I,b_i}(\mathbb{Q}_p)} \times X_w^{M_I}(b_i)$$

where M_I = Levi factor of P_I.

Cor. If $[b]_G \cap P_{\overline{\mathrm{supp}(w)}} = \emptyset$, then $X_w(b) = \emptyset$.

└ σ-st. closure of supp(w); min. rat. parabolic cont. B, i.e.

3) Frob. cyclic shift.

Def. $w, w' \in W$ related by Frob. cyclic shift $(w \overset{\sigma}{\leftrightarrow} w')$, if $\exists\ w = w_1, w_2, \dots, w_n = w'$,

s.t. $\forall i$: $\exists\ w_i = v_1 v_2$, $w_{i+1} = v_2 \sigma(v_1)$ & $\ell(w) = \ell(v_1) + \ell(v_2) =$
$= \ell(w')$.

Thm. (Geck, Kim, Pfeiffer, He) $C \subseteq W$ σ-conj. class,

$C_{min} \subseteq C$ min. el-ts. If supp$(w) = S$ for some (any) $w \in C_{min}$,

then $\forall\, w, w' \in C_{min}$: $w \overset{\sigma}{\leftrightarrow} w'$.

Prop. $w \overset{\sigma}{\leftrightarrow} w' \Rightarrow \forall b$: $X_w(b) \cong X_{w'}(b)$.

Pf: same arg. as in classical DL.

⑤ Ind. representability

Idea: adopt proof of Bonafé-Rouquier. of "X_w affine" to our setup.

Braid Monoid:

$$ {}_{\sigma}\circlearrowright B^+ = \left\langle (\underline{x})_{x\in W} \;\middle|\; \forall x,x'\in W: \ell(xx') = \ell(x)+\ell(x') \Rightarrow \underline{x}\,\underline{x'} = \underline{xx'} \right\rangle $$

<u>Thm.</u> $I \subseteq S$ σ-stable, $w \in W_I$ s.t. $\exists\, d>0$ & $a \in B^+$ with $\underline{w}\,\sigma(\underline{w})\cdots\sigma^{d-1}(\underline{w}) = \underline{w_I}\,.a$. Then $X_w(b)$ ind-scheme.

└ w_I longest in W_I. (〃 for $\dot{X}_{\dot{w}}(b)$)

<u>Proof.</u> ⊛ By Thm on p. 12: wlog. $I = S$.

⊛ For $x_1, \dots, x_r \in W$, let

$$ O(x_1,\dots,x_r) := O(x_1) \times_{G/B} O(x_2) \times_{G/B} \cdots \times_{G/B} O(x_r) $$

$$ = \left\{ g_1 \xrightarrow{x_1} g_2 \xrightarrow{x_2} \cdots \xrightarrow{x_r} g_{r+1} \right\} \subseteq (G/B)^{r+1}. $$

⊛ (Deligne) If $y_1, \dots, y_s \in W$ s.t.

$\underline{x_1}\cdots\underline{x_r} = \underline{y_1}\cdots\underline{y_s} \in B^+$, then $O(x_1,\dots,x_r) \cong O(y_1,\dots,y_s)$,

even canonically.

<u>Expl.</u> type A_2 : $\underline{sts}.\underline{sts} = \underline{sts}.\underline{tst} = \underline{st}.\underline{s}.\underline{t}.\underline{st} = \underline{st}.\underline{st}.\underline{st}.$

$W = \langle s,t\rangle$

$$\left\{\begin{tikzcd} g_1 \arrow[rr, "sts"] \arrow[dr, "st"'] & & g_2 \arrow[rr, "sts"] \arrow[dr, "t"] & & g_3 \\ & h_1 \arrow[ur, "s"'] & & h_2 \arrow[ur, "st"'] & \\ & \text{unique} & & & \end{tikzcd}\right\} \cong \left\{\begin{tikzcd} g_1 \arrow[r, "st"] & h_1 \arrow[r, "st"] & h_2 \arrow[r, "st"] & g_3 \end{tikzcd}\right\}.$$

Using this:

$w_0 \in W$ longest elt.

<u>Prop</u>. $x_1, \dots, x_r \in W$. If $v \in B^+$ s.t. $x_1 \cdots x_r = \underline{w_0} \cdot v$, then $\mathcal{O}(x_1, \dots, x_r)$ affine

<u>Pf</u>: $\mathcal{O}(x_1, \dots, x_r) \cong \mathcal{O}(w_0, \cdots)$ affine: generalizes the fact, that $\mathcal{O}(w_0) = G/T$ affine. $\square$

Now can rewrite $X_w(b)$ as

$$\begin{tikzcd} X_w(b) \arrow[r, "\text{cl.im.}"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \overbrace{L\mathcal{O}(w, \sigma(w), \dots, \sigma^{d-1}(w))}^{\text{ind-scheme, as } \mathcal{O}(-) \text{ affine}} \arrow[d] \\ L(G/B) \arrow[r, "\Delta"'] & L(G/B)^{d+1} \end{tikzcd}$$

$$s \longmapsto (s, b\sigma(s), \dots, (b\sigma)^{d-1}(s))$$

(if $g \in L(G/B)$ in $X_w(b)$, so $g \xrightarrow{w} b\sigma(g)$, so

$g \xrightarrow{w} b\sigma(g) \xrightarrow{\sigma(w)} (b\sigma)^2(g) \xrightarrow{\sigma^2(w)} (b\sigma)^3(g) \longrightarrow \dots \xrightarrow{\sigma^{d-1}} (b\sigma)^d(g)\ )$

Cartesian & Δ_d rep. by cl. im's. $\Rightarrow$ $X_w(b) \hookrightarrow L\mathcal{O}(\dots)$ too.

Cor. w of min lg. in its σ-conj. class, then $X_w(b)$ (& any $\dot{X}_{\dot{w}}(b)$ over it) is an ind-scheme.

Pf. Wlog $\overline{\mathrm{supp}(w)} = S$; using F-cyclic shift: enough for some $w' \in C_{min} \subseteq C = \sigma$-conj. class of w.

In C_{min} always ex. a "good" el-t, to which above Thm. applies. □
($\hookrightarrow$ Geck, Michel, Kim, Pfeifer, He)

Fun fact:

Prop. $b \in G(\breve{\mathbb{Q}}_p)$, $C \subseteq W$ σ-conj. class; if for some (any) $w' \in C_{min}$: $X_{w'}(b) \neq \emptyset$, then $\forall w \in C \setminus C_{min}$, $X_w(b)$ not a scheme.

In part., $\exists$ cases when $X_w(b)$ ind-sch, but not sch; eg. when $w = w_0$ & G not of type $A_1 \times \dots \times A_1$.

(Idea: one can embed $L\mathbb{A}^1 \xhookrightarrow[\text{cl. im.}]{} X_w(b)$).

Rem. $G = GL_3$, $\dot{X}_{\dot{w}_0}(1)$ quite complicated.

⑤ Torsors $\dot{X}_{\dot{w}}(b) \to X_w(b)$.

conn. red. model of T.

$$\tilde{W} = N_G(T)(\breve{\mathbb{Q}}_p)/\mathcal{T}(\breve{\mathbb{Z}}_p) \cong X_*(T) \rtimes W \quad \text{ext. aff. Weyl gp. of } T.$$

To capture Frobenius (if G non-split):

$$\begin{tikzcd} \tilde{W} \rtimes \langle \sigma \rangle \arrow[r, two heads] & W \rtimes \langle \sigma \rangle \\ F_w \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, mapsto] & w\sigma \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

discrete space

$\circlearrowright$

$X_*(T)$ acts by conj. $\rightsquigarrow F_w/X_*(T)$ trivial $X_*T_{\langle \sigma_w \rangle}$-torsor

$\sigma_w = Ad(w) \circ \sigma$

Prop. $\exists$ can. map

$$\alpha_{w,b} : X_w(b) \longrightarrow \underline{F_w/X_*(T)}.$$

For $\dot{w} \mapsto w$ in $N_G(T)(\breve{\mathbb{Q}}_p)$ with image $\bar{\dot{w}} \in F_w/X_*T$,

$$\dot{X}_{\dot{w}}(b) \longrightarrow X_w(b)_{\bar{\dot{w}}} := \alpha_{w,b}^{-1}(\underline{\{\bar{\dot{w}}\}})$$

is a v-torsor under $\underline{T_w(\mathbb{Q}_p)}$ (where $T_w = \mathbb{Q}_p$-form of T

split over $\breve{\mathbb{Q}}_p$, with Frob. $Ad(w)\circ\sigma$.) ;

Rem: If $\mathbb{Q}_p$ repl. by $\mathbb{F}_p((t))$: even pro-étale torsor (using Bouthier – Cesnavicius); in mixed char case: to prove that pro-ét, need an improvement of v-descent results of Rydh.

Rem. ① $\mathrm{Im}(\alpha_{w,b})$ can be described explicitly (at least for w Coxeter)

② Essentially it corresponds to rat'l conj. classes of tori lying in the stable class corr. to w:

$W/\sigma\text{-conj} \simeq$ (stable conj. cl. of unram. max. tori) ; each stable class ∃ ∃ fin. many rat'l classes

(already for SL_2: two rat'l classes in one stable class)

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⑤ Example: $G = GL_n$, w Cox; b basic ($b=1$)

(jt. Chan.)

<u>Prop.</u> $X_w(b)$, $\dot{X}_{\dot{w}}(b)$ schemes,

$$\dot{X}_{\dot{w}}(b) \neq \emptyset \iff \mathrm{ord}_p(\det(\dot{w})) = \mathrm{ord}_p(\det(b)).$$

In that case: (b=1)

$$\underline{G(\mathbb{Q}_p)} \times \underset{\underline{\simeq \mathbb{Q}_{p^n}^\times}}{\underline{T_w(\mathbb{Q}_p)}} \circlearrowright \dot{X}_{\dot{w}}(1) \simeq \coprod_{G(\mathbb{Q}_p)/G(\mathbb{Z}_p)} g.X_{\mathbb{Z}_p}$$

$$X_{\mathbb{Z}_p} \cong \{ v \in L^+\mathbb{G}_a^n \mid \det(v|\sigma(v)|\dots|\sigma^{n-1}(v)) \in \mathbb{Z}_p^\times \}.$$

Write $L := \mathbb{Q}_{p^n} / \mathbb{Q}_p$,

$\circlearrowleft$
Γ Gal. gp

$$\theta \mapsto \pm R_T^G(\theta) := R\Gamma_c(\dot{X}_{\dot{w}}(b), \overline{\mathbb{Q}}_\ell)$$

$$\begin{tikzcd}
\boxed{\text{smooth char's of } L^\times = T(\mathbb{Q}_p) \text{ with trivial } \Gamma\text{-stabilizer}} / \Gamma \arrow[d, "\simeq"'] \arrow[dr, bend left, "\text{Howe's constr.}"] \arrow[drr, bend left] & & \\
\boxed{\begin{array}{l} n\text{-dim. irrep's } \sigma \\ \text{of } W_{\mathbb{Q}_p} \text{ s.t.} \\ \sigma \cong \sigma \otimes (\varepsilon \circ \mathrm{rec}_{\mathbb{Q}_p}) \end{array}} / \cong \arrow[r, "\sim", "LL"'] & \boxed{\begin{array}{l} \text{smooth supercusp.} \\ \text{irreps } \pi \text{ of} \\ GL_n(\mathbb{Q}_p) \text{, s.t.} \\ \pi \cong \pi \otimes (\varepsilon \circ \det) \end{array}} / \cong \arrow[r, "\sim", "JL"'] & \boxed{\begin{array}{l} \{ \\ \text{of } G_b(\mathbb{Q}_p) \\ \; (\varepsilon \circ \mathrm{Nrd}) \end{array}} / \cong
\end{tikzcd}$$

$$\theta \mapsto \sigma_\theta := \mathrm{Ind}_{W_L}^{W_{\mathbb{Q}_p}} \left(W_L \to W_L^{ab} \xrightarrow[\mathrm{rec}]{\sim} L^\times \xrightarrow{\theta} \overline{\mathbb{Q}}_\ell^\times \right) \cdot \mu$$

$$\sigma_\theta \longmapsto \pi_\theta^{GL_n} \longmapsto \pi_\theta$$

$\varepsilon : \mathbb{Q}_p^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ char.

s.t. $\ker(\varepsilon) = \mathrm{im}(\mathrm{Nm}_{L/\mathbb{Q}_p})$

$\mu : L^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ rectifier

$\mu|_{U_L} = 1, \; \mu(p) = (-1)^{n-1}.$

Thm. (Chan-I.'19,20) If $p>n$ & $\theta|_{U_L^1}$ has triv. Γ-stab., then

$$\pm R_T^G(\theta) \cong \pi_\theta = JL(LL(\sigma_\theta)).$$

In part., $\pm R_T^G(\theta)$ irred supercusp., $\theta \mapsto \pm R_T^G\theta$ inj./Γ.

Rem: 1) ass. on θ much weaker than "$\theta|_{U_L^h}=1$ & $\theta|_{U_L^{h-1}}$ has triv. Γ-stab" (=: "θ primitive/regular. $\Leftrightarrow$ Howe dec. of θ has 1 member.)

2) rectifier is not captured; but JL is:

$$\begin{tikzcd}
\theta \arrow[r, mapsto] \arrow[dr, <->] & \pm R_T^{GL_n}(\theta) \arrow[d, <->, "JL"] \\
& \pm R_T^{\text{inner form}}(\theta)
\end{tikzcd}$$

(there is a sign change in JL)
it's captured by $\pm$'signs of $R_T^{\cdots}$

3) sketch of proof: by using DL-Methods:

$$\pm R_T^G(\theta) \underset{①}{=} c\text{-}\mathrm{Ind}_{ZG(\mathbb{Z}_p)}^{G(\mathbb{Q}_p)}(\text{irrep.}) \underset{②}{=} \bigoplus_{\text{fin.}} \begin{pmatrix}\text{irred.}\\ \text{supercusp.}\end{pmatrix}$$

+ ③ : compute formal degree (= dim of this)

Then (non DL):

④ compare with $JL(LL(\sigma_\theta))$ by matching some traces + f.deg.