The ramification of p-adic representations coming from geometry

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Overview of ramification theory: Riemann surfaces

- Let $f : Y \to X$ be a 'Galois' holomorphic map of comapct Riemann surfaces let $y \in Y$ and $f(y) = x$.
- Let t be a parameter at x. Then we may choose a parameter u at y such that $u^{e_x} = t$. We call e_x the *inertia degree* of f at y.
- The e_x are the 'error' terms in the Riemann-Hurwitz formula: $\chi(Y) = \chi(X) + \sum e_x - 1$.
- Let $Deck_y \subset Deck(Y/X)$ be the deck transformations fixing y. Then

$$\begin{aligned} Deck_y &\to \mathbb{C}^\times \\ g &\mapsto \frac{g(u)}{u} \end{aligned}$$

  is an injective character.
- The key observation is that $|Deck_y|$ is invertible in the ring extension $\mathbb{C}[[t]] \to \mathbb{C}[[u]]$.

Overview of ramification theory: classical arithmetic situation

- Now let F be $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$ and let K/F be a finite separable Galois extension (this arises by localizing extensions of number fields or covers of curves over $\mathbb{F}_q$).
- If $p \nmid |I(K/F)|$, things look like they did over $\mathbb{C}$. We call this *tame*.
- Otherwise, things get *wild*. We want to measure how wild $g \in G(K/F)$ is.

Overview of ramification theory: classical arithmetic situation, an example

- Let u be a solution to $X^p - t^{pn-1}X + t = 0$ over $\mathbb{F}_q((t))$.
- The conjugates of u are $u + it^n$ where $i = 1, \ldots, p-1$.
- Large n means $\mathbb{F}_q((u))$ is wilder over $\mathbb{F}_q((t))$.

Overview of ramification theory: classical arithmetic situation

- Let u be a uniformizer of K. Write $g(u) - u = \sum a_i u^i$. Let a_λ be the first nonzero coefficient.
- When tame, $g(u) = \zeta u$ for some root of unity ζ, and thus $\lambda = 1$.
- The bigger λ is, the more wild the action of g is.

Overview of ramification theory: the ramification filtration

- We may define a decreasing filtration $G(K/F)^s$ on $G(K/F)$.
- If $s > r$, the elements of $G(K/F)^s$ are wilder than those of $G(K/F)^r$.
- For $s \gg 0$, we have $G(K/F)^s = \{0\}$.
- This filtration is well behaved with quotients. Thus, we obtain a ramification filtration on $G(F^{sep}/F)$.
- Let $s(K/F)$ be the smallest s such that $G(K/F)^{s+\epsilon} = \{0\}$.

Overview of ramification theory: higher dimensions

- Let F be a discrete valuation field in char $p > 0$ and let K/F be a finite separable Galois extension.
- Abbes and Saito defined a ramification filtration $G(K/F)^s$.
- The residual extension need not be separable.
- Used to study ramification on higher dimensional varieties.

Overview of ramification theory: higher dimensions

- Let X be a variety over a perfect field k. Let $E = \sum E_i$ be a reduced Cartier divisor and $U = X - E$.
- Let $V \to U$ be an étale Galois cover. By localizing at each E_i we obtain extensions of discrete valuation fields K_i/F_i.
- The conductor divisor of $V \to U$ is defined to be $\sum s(K_i/F_i)[E_i]$.

Overview of ramification theory: a higher dimensional example

- Consider $\mathbb{A}^2 = \mathrm{Spec}(k[x, y])$ and let E be the x-coordinate cut out by the equaiton $y = 0$.
- The discrete valued field is then $F = k(x)((y))$ with the y-adic valuation. The residue field is $k(x)$.
- Consider the cover $V \to \mathrm{Spec}(k[x, y, y^{-1}])$ given by $Z^p - Z = \frac{x}{y^{pn}}$. This gives rise to an extension K of F.
- The residual extension of K/F is $k(x^{\frac{1}{p}})/k(x)$, i.e. imperfect.
- The conductor divisor is $pn[E]$.

p-adic representations and p-adic Lie filtrations

- Let $\rho : G(F^{sep}/F) \to GL_n(\mathbb{Z}_p)$ be and let $G = Im(\rho)$.
- There is a decreasing p-adic Lie filtration:

$$G(n) = \ker(\rho \mod p^n).$$

- This gives a p-adic Lie tower of field extensions F_n/F.
- Similarly, if we have a p-adic representation of $\pi_1^{et}(U)$, we obtain a tower of étale covers $U_n \to U$.

Question

Is there any relation between the p-adic Lie filtration on G and the ramification filtration? Alternatively, how do the conductors $s(F_n/F)$ grow along the tower?

p-adic representations and p-adic Lie filtrations: Sen's theorem

Theorem (Sen)

Let F be a finite extension of $\mathbb{Q}_p$ with ramification degree e and let $\rho : G(F^{sep}/F) \to GL_n(\mathbb{Z}_p)$ be a totally ramified continuous representation. Then there exists $c > 0$ such that $G^{en-c} \subset G(n) \subset G^{en+c}$. In particular, there exists $c_0 > 0$ such that

$$en - c_0 \leq s(F_n/F) \leq en + c_0.$$

p-adic representations and p-adic Lie filtrations: Sen's theorem

- $GL_n(\mathbb{Z}_p)$ is built up from "abelian pieces".
- Bound ramification of abelian pieces using class field theory.
- Patch together the information using the Herbrandt function.

p-adic representations and p-adic Lie filtrations: Sen's theorem

Question

Is there a similar result for $F = \mathbb{F}_q((t))$?

- Sen's method gives
$$s(F_n/F) \gg p^{n(1-\epsilon)}.$$
- There is no upper bound. The $s(F_n/F)$ can grow arbitrarily fast.
- If ρ is a character, then $s(F_n/F) \geq ps(F_{n-1}/F)$.

p-adic representations and p-adic Lie filtrations: A ray of hope from Benedict Gross

Theorem (Gross)

Let $F = \mathbb{F}_q((t))$ and assume that $\rho : G(F^{sep}/F) \to \mathbb{Z}_p^\times$ comes from a height h one dimensional formal group. Then there exists c such that $s(F_n/F) = cp^{hn}$.

- There are similar computations on the Igusa tower found Katz-Mazur's book on moduli of elliptic curves.
- We are unaware of other results.

p-adic representations and p-adic Lie filtrations: A ray of hope from Benedict Gross

Question

Does an analogue of Sen's theorem hold for "geometric" p-adic representations?
What should geometric mean?

Main results: local one dimensional

- Let $F = k((t))$ where k is perfect.
- Let $V \to \mathrm{Spec}(F)$ be an *ordinary* smooth proper variety.
- Let ρ be the $G(F^{sep}/F)$-representation associated to $H^i_{et}(X/F, \mathbb{Z}_p)$.

Theorem (K.)

One of the following holds:

1. ρ *has finite monodromy (i.e. image of inertia is* $< \infty$*).*
2. *There exists* $d > c > 0$ *such that for all* $n \geq 1$*:*

$$dp^n > s(F_n/F) > cp^n.$$

Main results: higher dimensional varieties

- Let X be a smooth variety over a perfect field k.
- Let E be a reduced divisor of X and set $U = X \backslash E$.
- Let $f : Y \to X$ be a smooth proper morphism such that Y_x is ordinary for $x \in U$ (in the sense of Bloch-Kato).
- Let ρ be the $\pi_1(U)$-representation be associated to $R^i_{et} f_* \mathbb{Q}_p$.

Theorem (K.)

Let E_i be an irreducible component of E and let F be the discrete valuation field associated to E_i. Consider the representation $\rho|_{G(F^{sep}/F)}$.

1. *$\rho|_{G(F^{sep}/F)}$ has finite monodromy.*
2. *Then there exists $d > c > 0$ such that*

$$dp^n \geq s(F_n/F) \geq cp^n.$$

Idea of proof: local F-crystals over $F = k((t))$

- We define a ring:

$$\mathcal{O}_{\mathcal{E}} := \left\{ \sum_{n=-\infty}^{\infty} a_n t^n \;\middle|\; \begin{array}{l} \text{We have } a_n \in \mathbb{Z}_p, \ \lim_{n \to -\infty} v_p(a_n) = \infty, \\ \text{and the } v_p(a_n) \text{ is bounded below.} \end{array} \right\},$$

- Let $\sigma : \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}$ be the map sending t to t^p.
- An F-crystal over $F = k((t))$ consists of:
  1. A free module V over $\mathcal{O}_{\mathcal{E}}$.
  2. An map $\varphi : \sigma^* V \to V$ that becomes an isomorphic after $\otimes \mathbb{Q}_p$.
  3. A connection $\nabla : V \to V \otimes \Omega_{\mathcal{O}_{\mathcal{E}}}$ (with some compatibility between φ and ∇).

Idea of proof: local overconvergent F-isocrystals over $F = k((t))$

- We define a subring of $\mathcal{E}$ of overconvergent functions:

$$\mathcal{O}_{\mathcal{E}^\dagger} := \left\{ f(t) \in \mathcal{O}_{\mathcal{E}} \,\middle|\, f(t) \text{ converges on an annulus } r < t < 1 \right\}.$$

- An overconvergent F-crystal over $F = k((t))$ is a convergent F-crystal whose structures descends to $\mathcal{O}_{\mathcal{E}^\dagger}$.
- Let $f : X \to \mathrm{Spec}(F)$ be smooth, proper and ordinary.
- By a theorem of Kedlaya, $R^i_{cris} f_* \mathbb{Z}_p$/torsion can be realized as an overconvergent F-crystal $M^\dagger$ over F.

Idea of proof: the unit-root subspace

- Let M^{u-r} be the largest subspace of $M^{\dagger} \otimes_{\mathcal{O}_{\mathcal{E}^{\dagger}}} \mathcal{O}_{\mathcal{E}}$ such that φ is an isomorphism.
- M^{u-r} is called the *unit-root* sub-crystal of $M^{\dagger} \otimes_{\mathcal{O}_{\mathcal{E}^{\dagger}}} \mathcal{O}_{\mathcal{E}}$.
- By the Riemann-Hilbert correspondence, M^{u-r} gives a p-adic representation ρ of $G(F^{sep}/F)$.
- ρ corresponds to $R^i_{et} f_* \mathbb{Z}_p$

Idea of proof: the unit-root subspace

Question

M^{u-r} sits inside of $M^{\dagger}$, which is overconvergent. What does this say about $\mathcal{M}^{u-r}$?

Question

How can we read the ramification of ρ from M^{u-r}?

Idea of proof: F-isocrystals with log decay

- We define a ring $\mathcal{O}_{\mathcal{E}^{log}}$ between $\mathcal{O}_{\mathcal{E}^\dagger}$ and $\mathcal{O}_{\mathcal{E}}$.
- The tails of elements in $\mathcal{O}_{\mathcal{E}^\dagger}$ decay linearly, e.g. $\sum p^n t^{-n}$.
- The tails of elements in $\mathcal{O}_{\mathcal{E}^{log}}$ decay like $\log_p$
- For example,

$$\sum p^n t^{-p^n}.$$

Proposition (K.)

The F-crystal M^{u-r}, which is a priori defined over $\mathcal{O}_{\mathcal{E}}$, descends to $\mathcal{O}_{\mathcal{E}^{log}}$.

Idea of proof: The monodromy of F-isocrystals with log decay

Question

How can we read the ramification of ρ from M^{u-r}?

Theorem (K.)

Let N be a unit-root F*-isocrystal over F (i.e. φ is an isomorphism). Let ψ be the corresponding representation of $G(F^{sep}/F)$ and let F_n'/F be the corresponding p-adic Lie tower. Then N descends to $\mathcal{O}_{\mathcal{E}^{log}}$ if and only if there exists $c > 0$ such that for all n:*

$$cp^n > s(F_n'/F).$$

Idea of proof: Finish the local proof

- ρ corresponds to $R^i_{et} f_* \mathbb{Z}_p$.
- ρ also corresponds to M^{u-r}.
- By the Proposition M^{u-r} descends to $\mathcal{O}_{\mathcal{E}^{log}}$.
- By the Theorem, we get an upper bound on conductors.
- For the lower bound: exterior power trick and abelian ramification theory.

Idea of proof: Higher dimensional case

- Theorem of Hu bounds Abbes-Saito ramification by cutting along curves.
- This requires "uniform" bounds of ramification along curves.
- Need higher dimensional notion of F-crystals with log-decay.

Further questions

Definition

The fierce degree of K/F is the inseperable degree of the residue fields.

Question

If ρ is geometric, can the fierce degree of F_n/F tend to infinity? Or is it finite?

Question

What about the p-adic monodromy of non-ordinary fibrations? We have partial results, but nothing as definitive.

Thanks for listening!