Endoscopic points on the SL(2)-eigencurve

Big goal: Understand Langlands[1] functoriality in the p-adic Langlands programme [2] later.

① In a nutshell:

G conn. red. gp $/\mathbb{Q}$ split $\rightsquigarrow$ $\widehat{G}/\overline{\mathbb{Q}}$ = dual group again conn. red.

⌜root datum $(X^*, \phi, X_*, \phi^\vee)$
flip it $\rightsquigarrow (X_*, \phi^\vee, X^*, \phi) \rightsquigarrow \widehat{G}$⌟

e.g. $\widehat{GL_n} = GL_n$
$\widehat{SL_n} = PGL_n$

$\widehat{G}$ is the target of Galois reps (L-parameters) associated to automorphic reps. of $G(\mathbb{A})$.

Now let G, H $/\mathbb{Q}$ split.

Langlands functoriality conjectures:

A morphism $\widehat{G} \longrightarrow \widehat{H}$ gives rise to a transfer

$$\left\{\begin{matrix}\text{L-packets for} \\ \text{autom. reps} \\ \text{of } G\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{L-packets of} \\ \text{aut. rep of} \\ H\end{matrix}\right\}$$

members have same Galois reps.

<u>Fact</u>: L-packets of Gl_n are singletons.

<u>Example</u>: (Labesse - Langlands transfer)

The natural map $\widehat{GL_2} = GL_2 \longrightarrow PGL_2 = \widehat{SL_2}$ corresponds to a transfer

$$\tilde{\pi} = \bigotimes \tilde{\pi}_\ell \longmapsto \Pi = \text{L-packet of reps of } SL_2(\mathbb{A})$$

autom. rep of Gl_2

$$= \bigotimes_\ell{}' \Pi_\ell$$

local L-packets.

<u>Local L-packets</u>.

- ℓ finite prime

  $\tilde{\pi}_\ell$ smooth irred. of $Gl_2(\mathbb{Q}_\ell)$

Note $GL_2(\mathbb{Q}_\ell) / Z(\mathbb{Q}_\ell) SL_2(\mathbb{Q}_\ell)$ finite abelian.

$\rightsquigarrow \quad \tilde{\pi}_\ell \big|_{SL_2(\mathbb{Q}_\ell)} \cong \bigoplus_{i=1}^{n} \pi_{\ell,i}$, $\pi_{\ell,i}$ distinct irred. & $n = \{1,2,4\}$.

The $\pi_{\ell,i}$'s form the local L-packet Π_ℓ associated to $\tilde{\pi}_\ell$.

- $\tilde{\pi}_\infty = D_k$ discrete series, wt $k \geq 2$.

$\Pi_\infty = \{ D_k^+ , D_k^- \}$ discrete series L-packet of size 2.

holomorphic antiholomorphic.

<u>Fact</u>: If $\tilde{\pi}_\ell$ unramified (i.e. $\tilde{\pi}_\ell^{GL_2(\mathbb{Z}_\ell)} \neq 0$) there is a unique $\pi_\ell^0 \in \Pi_\ell$ s.th. $\pi_\ell^{SL_2(\mathbb{Z}_\ell)} \neq 0$.

$$\Pi := \left\{ \pi = \bigotimes_\ell \pi_\ell \;\middle|\; \pi_\ell \in \Pi_\ell \ \& \ \pi_\ell = \pi_\ell^0 \text{ for almost all } \ell \right\}.$$

2 kinds of packets: . Π stable, i.e.

$\forall \pi \in \Pi$, $m(\pi) = 1$

multiplicity in the autom. spectrum.

- Π unstable / endoscopic :

  arise as transfer of $\tilde{\pi}$ with complex multiplication.

  If $\pi \in \Pi$ then $m(\pi) = \{0, 1\}$ &

  roughly half of the π's are automorphic, i.e. satisfy $m(\pi) = 1$.

Fix Π endoscopic, s.th. $\Pi_\infty = \{D_k^+, D_k^-\}$

$k \geq 2$.

Fix $\tau \in \Pi$ s.th $\tau_\infty = D_k^+$

$\tau = \bigotimes \tau_\ell$ & τ automorphic.

Then $\tau^- := \bigotimes_{\ell \text{ finite}} \tau_\ell \otimes D_k^-$ has $m(\tau^-) = 0$.

τ^- is an adm. rep. of $SL_2(\mathbb{A})$.

Quest : Find τ^- in the p-adic world!

② SL(2) - eigencurves

- Eigenvarieties are adic spaces interpolate systems of Hecke eigenvalues of automorphic forms p-adically.

For SL(2) can be constructed using either

Ⓐ completed cohomology (Emerton)

Ⓑ overconvergent cohomology (Hansen)

Ⓐ $K = K^p K_p \subset SL_2(\mathbb{A}_f)$
compact open

$$Y(K) = SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}) / K\, SO_2(\mathbb{R}).$$

$$\tilde{H}^1(K^p) := \varprojlim_s \varinjlim_{K_p} H^1(Y(K_pK^p), \mathbb{Z}/p^s\mathbb{Z}) \otimes \mathbb{Q}_p$$

$\circlearrowleft$ $\mathbb{T}^{ur}$ Hecke-algebra

"space of p-adic automorphic forms".

finite slope construction

$\rightsquigarrow$ Jacquet-module

$$\begin{tikzcd} \mathcal{E}(K^p) \arrow[d, "\text{locally finite on the source.}"'] \\ \mathcal{W} \end{tikzcd}$$

- eigencurve of tame level K^p
- an adic space

pts $\longleftrightarrow$ systems of Hecke eigenvalues of finite slope p-adic automorphic forms

weight space
finite disjoint union of open unit discs.

Notation:

$$K^p = K^{ram} \cdot K^{ur}$$
$$= \prod_{\ell \in S(K^p)} K_\ell \cdot \prod SL_2(\mathbb{Z}_\ell).$$

$\mathcal{E}(K^p)$ comes equipped with coherent sheaf $\mathcal{M}$ of p-adic forms

If $z \in \mathcal{E}(K^p)(\overline{\mathbb{Q}}_p)$ is a classical pt.

i.e. $z \longleftrightarrow \lambda : \mathbb{T} \longrightarrow \overline{\mathbb{Q}}_p$ system of Hecke EV's appearing in a classical cusp. aut. rep.

$$\exists \quad 0 \neq M_{\bar z}^{cl} \subset M_{\bar z}$$ subspace of classical forms.

concretely:

$$M_{\bar z}^{cl} = \mathrm{Hom}\Big(\underbrace{J_B}_{\text{Jacquet-module}} \big(\underbrace{H^1(K^p, \underbrace{\mathrm{Sym}^k(\mathbb{Q}_p^2)}_{W})}_{\cong\, \varinjlim_{K_p} H^1(Y(K^pK_p),\, W)} \big)^{\overset{\lambda\text{-eigenspace}}{\lambda}}, \underbrace{k(z)}_{\text{res. field}} \Big)$$

$$\& \quad J_B(H^1(K^p, W))^{\lambda} \otimes_2 \mathbb{C}$$

$$\cong \bigoplus_{\pi \text{ adm reps}} m(\pi) \left(\pi_{S(K^p)}\right)^{K^{ram}} \otimes J_B(\pi_p)^{\lambda_p} \otimes H^1_{rel.lie}(g, K_\infty, W_{\mathbb{C}} \otimes \pi_\infty)$$

$\pi_\ell = \pi(z)_\ell$

$\forall\, \ell \notin S(K^p) \cup \{p\}$

$$\lambda = \lambda^{ur} \otimes \lambda_p$$

- It's an interesting question to study the difference between $M_{\bar z}^{cl}$ & $M_{\bar z}$.

If z is of non-critical slope then equality $M_{\bar z}^{cl} = M_{\bar z}$ (Coleman Classicality).

Recall Π endoscopic, $\tau \in \Pi$ $\quad S = \{\ell \text{ primes} \mid \Pi_\ell \text{ ramifies}\}$

$\tilde{\tau}$ $\qquad\qquad\qquad\qquad \tau^-$

Assume
- $\tilde{\tau}$ has CM by F, p splits in F
- Π_p is singleton & unramified
- $\ell \in S \rightsquigarrow \Pi_\ell$ = supercuspidal L-packet of size 2
- $2 \notin S$.

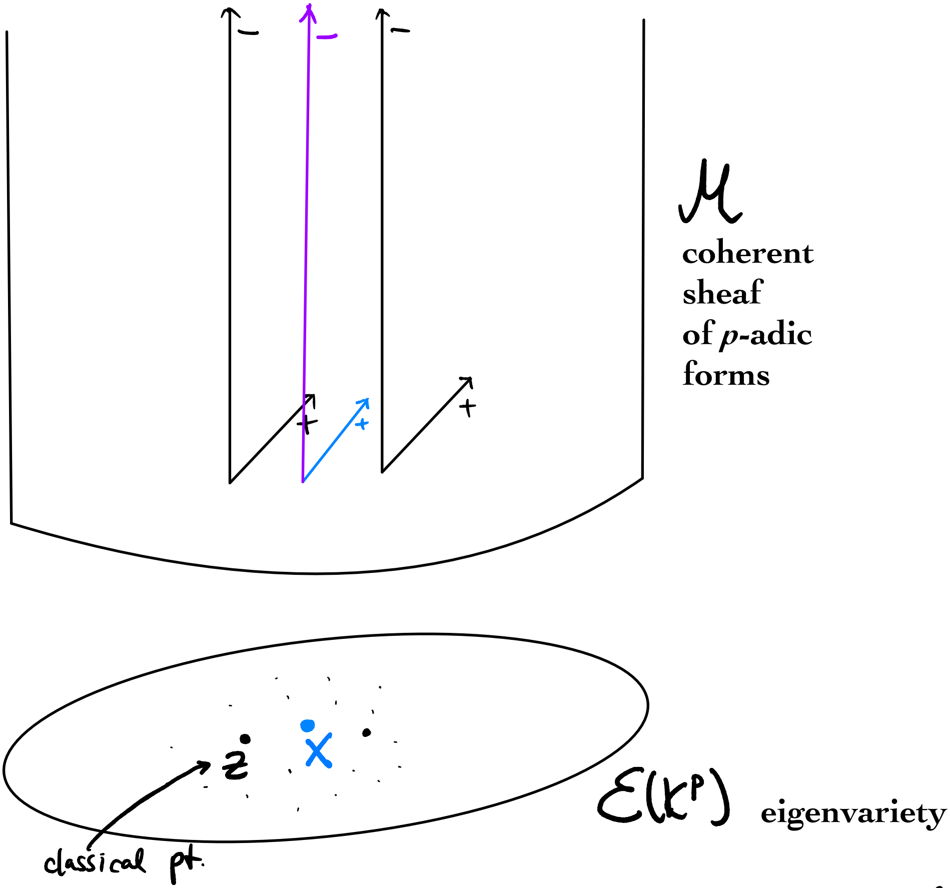
Fix K^p "minimal" s.t. $(\tau_f^p)^{K^p} \neq 0$.

$\tau \rightsquigarrow x$ classical pt of critical on $\mathcal{E}(K^p)$.

Thm (L.) For x as above we have

$$\mathcal{M}_{\bar{x}} / \mathcal{M}_{\bar{x}}^{cl} \neq 0.$$

Proof: (1) In small nbhd. of x all classical pts are stable.



② Stable packets contribute twice as much to fibre as endoscopic pts:

π stable then both reps

π & $\pi' = \bigotimes \pi_\ell \otimes D_\ell$ the other one

will contribute.

need to control $\dim\left(\pi_{S(K^p)}\right)^{K^{ram}}$.

③ The fibre rank of coherent sheaf is upper semicont., i.e. the dim cannot drop at x. ∎

Remarks.

Thm $\Rightarrow$ $\exists$ f p-adic eigenform with same Hecke EV's as a classical form $g \in \tau$.

The eigencurve carries a family of projective Galois reps & f and τ have same proj. Galois rep.

Hypothetical global p-adic L-packet that contains τ also contain (the rep. generated by) f.

Open questions:

- local geometry around x:
  Is $\mathcal{E}(K^p)$ smooth at x?
- In above story f is only implicitly connected to τ. Is there an explicit connection?

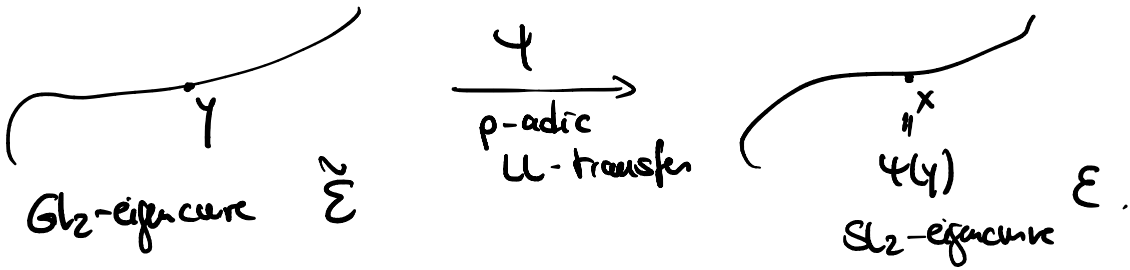
Ongoing work w. C. Johansson:

Explain the Sl_2-picture via Gl_2 & answer these questions.

We work w. overconvergent cohomology.

$\tilde{\pi}$ aut. rep. w. CM giving to Π

$\rightsquigarrow$ y critical slope refinement.



As $\tilde{\pi}$ has CM $\exists$ χ quadr. char. of $\mathbb{A}^\times$ s.th.

$$\tilde{\pi} \cong \tilde{\pi} \otimes \chi$$

Twisting works on the eigencurve, i.e. have an action of $\mathbb{Z}/2\mathbb{Z} = \{1, \chi\}$ on $\tilde{\mathcal{E}}$ that sends classical pt $\leftrightsquigarrow \tilde{\pi}$

to pt $\leftrightsquigarrow \tilde{\pi} \otimes \chi$.

Note y is a fixed pt,

y is isolated amongst CM forms.

We show: locally around x & y,

$\mathcal{E}$ is the quotient of $\tilde{\mathcal{E}}$ by $\mathbb{Z}/2\mathbb{Z}$.

Bellaiche $\Rightarrow$ $\tilde{\mathcal{E}}$ is smooth at y.

$$\text{so} \quad \widehat{\mathcal{O}}_{\tilde{\mathcal{E}},y} \cong E[[t]]$$

appropriate p-adic base field.

& $\chi \in \mathbb{Z}/2\mathbb{Z}$ sends $t \mapsto -t$.

in particular
$$\widehat{\mathcal{O}}_{\mathcal{E},x} \cong E[[t]]^{\mathbb{Z}/2\mathbb{Z}}$$
$$\cong E[[t^2]] \text{ is smooth.}$$

In fact we work with SL_2/F, F tot. real.
and smoothness is the exception.