p-adic étale cohomology of period domains

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Main interest: Cohomology of p-adic symmetric spaces

$$\begin{tikzcd}
\text{moduli spaces of weakly admissible} \\ \text{filtrations on isocrystals} \arrow[d, equal] & \\
p\text{-adic period domains} \arrow[r, hook] \arrow[d, "\text{classical analog}"] & p\text{-adic symmetric spaces} \\
\text{moduli spaces of semistable vector} \\ \text{bundles on a Riemann surface} &
\end{tikzcd}$$

Foundations: **van der Put-Voskuil, Rapoport-Zink, ..., Dat-Orlik-Rapoport**

Example

- $\Omega^d_{\mathbf{Q}_p}$ – Drinfeld space, $\mathbb{P}_{\check{\mathbf{Q}}_p}, \mathbb{A}^{ad}_{\mathbf{Q}_p}$.
- the corresponding groups:

$$\mathbb{GL}_{d+1}(\mathbf{Q}_p), \quad D^{\times}_{\check{\mathbf{Q}}_p} \text{- division algebra over } \check{\mathbf{Q}}_p, \quad \mathbf{Q}_p^* \times \mathbf{Q}_p^*.$$

Example: Drinfeld symmetric spaces

Notation: K-cdvf, $(0,p)$, $K \supset \mathscr{O}_K \to k$, k perfect, $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$, $C = \widehat{\overline{K}}$. Fix ℓ-prime.

Definition Drinfeld symmetric space of dimension d:

- $\Omega^d_K = \mathbb{P}^d_K \setminus \cup_{H \in \mathscr{H}} H$, $\mathscr{H}$ - set of K-rational hyperplanes;
- have action of $G = \mathbb{GL}_{d+1}(K)$.

(A) Étale cohomology

Theorem (Schneider-Stuhler, Colmez-Dospinescu-N) There is a $G \times \mathscr{G}_K$-equivariant isomorphism

$$H^i_{\text{ét}}(\Omega^d_C, \mathbf{Z}/\ell^n) \simeq \mathrm{Sp_i}(\mathbf{Z}/\ell^n)^*(-i),$$

where $\mathrm{Sp_i}(\mathbf{Z}/\ell^n)$ is the generalized smooth Steinberg representation for parabolic with blocks $(d+1-i) \times i$.

Proof.

- cases $\ell \neq p$ (SS) and $\ell = p$ (CDN) have different proofs
- CDN:
  1. use integral p-adic Hodge Theory of Bhatt-Morrow-Scholze, Česnavičius-Koshikawa
  2. étale $\Rightarrow$ de Rham and crystalline computations: use the strong version of ordinarity of the standard formal semistable model of Ω_K^d.

$\square$

Now we assume $K = \mathbf{Q}_p$ (by Weil descent $\Rightarrow$ $K/\mathbf{Q}_p$ – finite).

(B) Étale cohomology with compact support

Theorem (Orlik, C-D-Hauseux-N) ℓ big enough. There is a $G \times \mathscr{G}_K$-equivariant isomorphism

$$H^i_{\text{ét},c}(\Omega^d_C, \mathbf{Z}/\ell^n) \simeq \mathrm{Sp}_{2d-i}(\mathbf{Z}/\ell^n)(d-i)$$

Proof.

Part 1: *geometric part*, same for all ℓ.

Part 2: *group theoretic part* – very different for $\ell \neq p$ and $\ell = p$. $\square$

(C) What happens ℓ-adically ?

Theorem (Orlik, CDHN) Let $\ell \geq 5$.

- continuous cohomology:

$$H^i_{\text{ét},c}(\Omega^d_C, \mathbf{Z}_\ell) \simeq \mathrm{Sp}^{\mathrm{cont}}_{2d-i}(\mathbf{Z}_\ell)(d-i)$$

- Huber cohomology:
  1. $\ell \neq p$:
  $$H^i_{\text{ét},c,Hu}(\Omega^d_C, \mathbf{Z}_\ell) \simeq \mathrm{Sp}_{2d-i}(\mathbf{Z}_\ell)(d-i)$$
  2. $\ell = p$: ????

Here, for X, adic space:

- continuous cohomology:

$$\mathrm{R}\Gamma_{\text{ét},c}(X, \mathbf{Z}_\ell) := \mathrm{R}\varprojlim_n \mathrm{R}\Gamma_{\text{ét},c}(X, \mathbf{Z}/\ell^n)$$

- Huber cohomology: X – partially proper:

$$\mathrm{R}\Gamma_{\text{ét},c,Hu}(X, \mathbf{Z}_\ell) := \mathrm{R}\Gamma_{\text{ét},c}(X, \mathrm{R}\Gamma\pi_*(\mathbf{Z}/\ell^n)),$$

where $\pi : (\mathscr{F}_n) \mapsto \varprojlim_n \mathscr{F}_n$.
For $\ell \neq p$ it was defined to pick up smooth representations but strange things happen for $\ell = p$:

$$\begin{aligned} (a) \quad & \ell \neq p: \quad H^2_{\text{ét},c,Hu}(\mathbb{A}_C, \mathbf{Z}_\ell(1)) \simeq \mathbf{Z}_\ell, \\ (b) \quad & \ell = p: \quad H^2_{\text{ét},c,Hu}(\mathbb{A}_C, \mathbf{Z}_p(1)) \simeq (\mathscr{O}_{\mathbb{P}^1_C,\infty}/C) \oplus \mathbf{Z}_p \end{aligned}$$

compatible with the action of $G = \mathbb{G}_m(\mathbf{Q}_p) \times \mathbb{G}_m(\mathbf{Q}_p)$, not smooth.

Remarks

(A) **Duality**: Have abstractly ($p \geq 5$)

$$H^i_{\text{ét},c}(\Omega^d_C, \mathbf{Z}/p^n) \simeq H^{2d-i}_{\text{ét}}(\Omega^d_C, \mathbf{Z}/p^n)^\vee$$

Q: Do we have a duality for partially proper adic spaces ??

($\bullet$) Proper case: $\leftarrow$ Offer Gabber, Bogdan Zavyalov.
($\bullet$) Almost proper case: rationally, $\leftarrow$ Lan-Liu-Zhu.

(B) **Where are the other Steinberg representations ??**

- Only certain generalized Steinberg representations appear.
- For $\ell \neq p$, Dat showed that $\mathrm{R}\Gamma_c$ "knows" about all Steinberg representations.

Main theorem

Theorem (CDHN) Let $(\mathbb{G}, [b], \{\mu\})$ be a local Shtuka datum with $\mathbb{G}/\mathbf{Q}_p$ quasi-split, $b \in \mathbb{G}(\breve{\mathbf{Q}}_p)$ basic and s-decent. Let $p \geq 5$. There are isomorphisms of $\mathscr{G}_{E_s} \times J(\mathbf{Q}_p)$-modules:

$$H^*_{\text{ét},c}(\mathscr{F}^{wa}_C, \mathbf{Z}/p^n) \simeq \bigoplus_{[w] \in W^\mu/\mathscr{G}_{E_s}} v^J_{I_{[w],n}}\{-\ell_{[w]}\}[-n_{[w]}],$$

$$H^*_{\text{ét},c}(\mathscr{F}^{wa}_C, \mathbf{Z}_p) \simeq \bigoplus_{[w] \in W^\mu/\mathscr{G}_{E_s}} v^{J,\text{cont}}_{I_{[w]}}\{-\ell_{[w]}\}[-n_{[w]}],$$

where $\{a\} := (a)[2a]$, $n_{[w]} = |\Delta \setminus I_{[w]}|$.

Notation

(1) $[b] \in B(G)$:

- σ conj. classes in $G(\breve{\mathbf{Q}}_p)$ or, equivalently,
- isomorphism classes of G-isocrystals;
- $[b] \mapsto N_b$
- Example: $G = GL_n$: $B(G) =$ isomorphism classes of n-dimensional isocrystals

(2) $\{\mu\}$:

- conjugacy class of geometric cocharacters,
- specifies the type of filtration on N_b.
- $[b] \in A(G, \mu)$: "Hodge of $\mu \leq$ Newton of N_b"

(3) $\mathscr{F}$:

- $\mathscr{F} = \mathscr{F}(G, \mu) = G_E/P(\mu)$ – flag variety$/E$,
- E the field of definition of μ;
- moduli spaces of filtrations of type μ

(4) $\mathscr{F}^{wa}$ – **period domain**:

- $\mathscr{F}^{wa} = \mathscr{F}^{wa}(G, [b], \{\mu\}) \subset \mathscr{F}$: locus, where the filtration is weakly admissible
- s-decent $\rightarrow \mathscr{F}^{wa}/E_s, E_s = E\mathbf{Q}_{p^s}$

(5) J_b:

- automorphism group of N_b: $J_b = \mathrm{Aut}(N_b)$
- b-basic iff $J = J_b$ – inner form of G
- $\mathscr{F}^{wa}$ has action of $J(\mathbf{Q}_p)$
- $J_b(\mathbf{Q}_p) = \{g \in G(\overline{\mathbf{Q}}_p) | gb\sigma(g^{-1}) = b\}$.
- Example:
  $\Omega^d_{\mathbf{Q}_p}, G = GL_{d+1}, [b] = [1], J = G, \{\mu\} = (d, -1, -1, \ldots, -1)$

- $S \subset J_{der}$ – max split torus,
- $P_0 \supset S$ – minimal parabolic,
- $T \supset S$ – max torus
- $\Delta = \{\alpha_1, \ldots, \alpha_d\} \subset X^*(S)$ – relative simple roots
- $\{\omega_{\alpha_i}\}$ – dual basis of 1-PS

- W – Weyl group of G wrt T (μ factors through T),
- W^μ – Kostant representative of $W/Stab(\mu)$
- $v^J_{I_{[w]},n}$ – generalized Steinberg (smooth), values in $\mathbf{Z}/p^n$

Proof of the main Theorem for $\mathbf{Z}/p^n$-coefficients

(1) **the geometric part**: $(\mathscr{F}^{wa}, \mathscr{F}, \partial\mathscr{F}^{wa})$, $\partial\mathscr{F}^{wa} : \mathscr{F} \setminus \mathscr{F}^{wa}$, proper pseudo-adic, $\to$

$$\mathrm{R}\Gamma_{\text{ét},c}(\mathscr{F}^{wa}_C) \to \mathrm{R}\Gamma_{\text{ét}}(\mathscr{F}_C) \to \mathrm{R}\Gamma_{\text{ét}}(\partial\mathscr{F}^{wa}_C)$$

Enough to show that (uses Bruhat cells + $\mathbb{A}_{alg}$ has trivial $\mathrm{R}\Gamma_{\text{ét}}$):

$$H^*_{\text{ét}}(\partial\mathscr{F}^{wa}) \simeq \bigoplus_{|\Delta\setminus I_{[w]}|=1} \iota^J_{I_{[w]},n}\{-\ell_{[w]}\} \oplus \bigoplus_{|\Delta\setminus I_{[w]}|>1} (\iota^J_{\Delta,n}\{-\ell_{[w]}\} \oplus v^J_{I_{[w]},n}\{-\ell_{[w]}\}[-|\Delta\setminus I_{[w]}|+1]) \tag{0.1}$$

Here, for $I \subset \Delta$ and $X_I = J(\mathbf{Q}_p)/P_I(\mathbf{Q}_p)$, we set

$$\iota^J_{I,n} := \iota^J_{P_I,n} = \mathrm{LC}(X_I, \mathbf{Z}/p^n).$$

Stratification of $\partial\mathscr{F}^{wa}$ by Schubert varieties

Recall: Schubert varieties have easy cohomology

Step 1 Idea: Faltings and Totaro:

weak admissibility=semistability condition

$\mathscr{F}^{wa}$ – locus, where semistability fails

Stratify:

$$\partial\mathscr{F}^{wa} = Z_1 \supset Z_2 \supset \dots Z_{i-1} \supset Z_i \supset Z_{i+1} \supset$$

Z_i – locus where semistability fails to degree $\geq i$.

Step 2: to make this precise:

use **Hilbert-Mumford criterium**:

K/E_s, $x \in \mathscr{F}(K)$ is semistable (i.e., $x \in \mathscr{F}^{wa}(K)$) iff

$$\mu(x, \lambda) \geq 0, \text{ for all } \lambda \in X_*(J)^{\mathscr{G}_{\mathbf{Q}_p}}.$$

Here $\mu(-,-)$ – the slope function associated to a linearization of the action of J (related to the metric on filtrations).

Remark **Simplified HM criterion**

- The slope function $\mu(x, -)$ is affine on each chamber of the spherical complex $\mathscr{B}(J_{der})$ (not just convex)
- hence it is enough to test λ's associated to the relative simple roots and their conjugates.

Notation:

- $\lambda \in X_*(J)_{\mathbf{Q}}$, Y_λ – locus in $\mathscr{F}$, where λ damages the ss condition $(\mu(-, \lambda) < 0)$
- $I \subset \Delta$, $Y_I := \cap_{\alpha \notin I} Y_{\omega_\alpha}$ – fundamental Schubert variety
- $Z_i := \cup_{|\Delta \setminus I| = i} Z_I$, $Z_I := X_I Y_I$, closed pseudo-adic

Have

$$\partial \mathscr{F}^{wa} = Z_1 = \cup_{|\Delta \setminus I| = 1} Z_I$$

Fundamental complex

Closed Mayer-Vietoris on $\partial\mathscr{F}_C^{wa} \to$ complex

$$0 \to \mathbf{Z}/p^n \to \oplus_{|\Delta\setminus I|=1}(\mathbf{Z}/p^n)_I \to \oplus_{|\Delta\setminus I|=2}(\mathbf{Z}/p^n)_I \to \cdots$$

Here $(\mathbf{Z}/p^n)_I$ – "evaluation" of $\mathbf{Z}/p^n$ on $Z_{I,C}$

Lemma This complex is acyclic.

Proof.

- Stalks are complexes of induced representations
- $\to$ get a complex of locally constant functions on a contructible subcomplex of $\mathscr{B}(J_{der})$.

$\square$

It follows that:

$$E_1^{i,j} = \bigoplus_{|\Delta \setminus I| = i+1} H^i_{\text{ét}}(\partial \mathscr{F}_C^{wa}, (\mathbf{Z}/p^n)_I) \Rightarrow H^{i+j}_{\text{ét}}(\partial \mathscr{F}_C^{wa})$$

Compute:

$$\begin{aligned} H^i_{\text{ét}}(\partial \mathscr{F}_C^{wa}, (\mathbf{Z}/p^n)_I) &\simeq \mathrm{LC}(X_I, H^i_{\text{ét}}(Y_{I,C})) \\ &\simeq \iota^J_{P_I} \otimes H^i_{\text{ét}}(Y_{I,C}) \\ &\simeq \iota^J_{P_I} \otimes \Big(\bigoplus_{[w] \in \Omega_I \subset W^\mu / \mathscr{G}_{E_s}} \mathbf{Z}/p^n(-\ell_{[w]})[-2\ell_{[w]}]\Big). \end{aligned}$$

Facts:

- This spectral sequence degenerates at E_2: use Galois weight argument
- The associated grading = RHS of the boundary formula

Remains: splitting of the grading

Splitting of the grading

(2) **Group theoretic part**: Enough to prove:

Theorem $p \geq 5$. Let $I, I' \subset \Delta$. Then

$$\mathrm{Ext}^1_J(v^J_{P_I}, v^J_{P_{I'}}) = 0, \text{ for } |(I \cup I') \setminus (I \cap I')| > 1.$$

Remark

- No idea how to compute higher Ext
- $\to$ no p-adic analog of Dat's computation of $\mathrm{R}\Gamma_c$.

Proof.

Difficulties (not present for $\ell \neq p$):

- Emerton's ordinary parts functor – not exact
- restriction to K-invariants – not exact

Idea:

- use ordinary parts functor + its derived functors to do devissage
- and reduce to the computation of $H^1(J(\mathbf{Q}_p), \mathrm{St}_J)$.

Thank you !