潘晗

2020/8/27

On locally analytic vectors of the completed cohomology of modular curves

Main tool: p-adic Hodge theory

Completed cohomology p

- $K \subseteq GL_2(\mathbb{A}_f)$

  $Y_K/\mathbb{Q}$, $Y_K(\mathbb{C}) = GL_2(\mathbb{Q}) \backslash \mathcal{H}^{\pm} \times GL_2(\mathbb{A}_f)/K$

- $K^p \subseteq GL_2(\mathbb{A}_f^p)$

Def'n $\tilde{H}^i(K^p, \mathbb{Z}_p) := \varprojlim_n \varinjlim_{K_p \subseteq GL_2(\mathbb{Q}_p)} H^i(Y_{K^pK_p}(\mathbb{C}), \mathbb{Z}/p^n)$

$\circlearrowleft GL_2(\mathbb{Q}_p)$

- $C = \widehat{\overline{\mathbb{Q}}_p} \supseteq \mathcal{O}_C$
- $\tilde{H}^i(K^p, \mathcal{O}_C) := \tilde{H}^i(K^p, \mathbb{Z}_p) \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C$
- $\tilde{H}^i(K^p, C) := \tilde{H}^i(K^p, \mathcal{O}_C) \otimes_{\mathcal{O}_C} C$

$GL_2(\mathbb{Q}_p) \circlearrowright$ Banach space $/C$

$\cup\!|$

$\mathfrak{g} \curvearrowright \tilde{H}^i(K^p, C)^{la} =$ $(GL_2(\mathbb{Q}_p))$-locally analytic vectors

- $\mathfrak{g} = C \otimes_{\mathbb{Q}_p} \mathrm{Lie}(GL_2(\mathbb{Q}_p)) = \mathfrak{gl}_2(C)$

- $B = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\} \subseteq GL_2(\mathbb{Q}_p)$
- $\mathfrak{b} = C \otimes \mathrm{Lie}(B)$
- $\mu_k : \mathfrak{b} \to C$ char, $k \in \mathbb{Z}$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto kd$$

- $\tilde{H}^i(K^p, C)^{la}_k$: μ_k-isotypic part

Determine $\tilde{H}^i(C)^{la}_k = \varinjlim_{K^p} \tilde{H}^i(K^p, C)^{la}_k$

as a rep'n of $GL_2(\mathbb{A}_f^p) \times \mathfrak{b} \times G_{\mathbb{Q}_p}$

$(G_{\mathbb{Q}_p} \curvearrowright C)$

Theorem 1 Assume $k \neq 1$.

$$\tilde{H}^1(C)^{la}_k = N_{k,1} \oplus N_{k,w}(k-1)$$

Hodge-Tate decomposition

Moreover,

(1) $k \neq 2$, $N_{k,w} = M^\dagger_{2-k} \otimes \chi_{k,w}$ ← a char of $GL_2(\mathbb{A}_f^p) \times \mathfrak{b}$

$M^\dagger_{2-k}$: overconvergent modular forms of wt $2-k$.

(2) $\underline{k \geq 2}$, $N_{k,1} = M_k^{\dagger}/M_k \otimes \chi_{k,1}$

M_k: classical modular forms of wt. k.

$Y_K \subseteq X_K$

$$M_k := \varinjlim_K H^0(X_{K,\mathbb{C}}, \omega^k)$$

$\underline{k < -1}$, $N_{k,1} = \dfrac{M_k^{\dagger}}{H^1(\omega^k)} \otimes \chi_{k,1}$ quot / sub

$0 \to H^1(\omega^k) \to N_{k,1} \to M_k^{\dagger} \to 0$ ext'n

$$H^1(\omega^k) := \varinjlim_K H^1(X_{K,\mathbb{C}}, \omega^k)$$

$$\varinjlim H^1(X_{K,\mathbb{C}}, \mathrm{Sym}^k \mathcal{O}) \xrightarrow{\theta} \tilde{H}^i(\mathbb{C})_k^{la} \xrightarrow{\theta^{k+1}} \tilde{H}^i(\mathbb{C})_{2-k}^{la}$$

$$\begin{array}{ccccc} \cup\!\! \mid & & \cup & & \\ M_{2-k} & \to & M_{2-k}^{\dagger} & \longrightarrow & M_{2-k}^{\dagger}/M_{2-k} \end{array}$$

$$M_k^{\dagger} := \varinjlim_K M_k^{\dagger}(K)$$

$K = K^p \Gamma(p^n)$ $\quad \Gamma(p^n) = 1 + p^n M_2(\mathbb{Z}_p)$

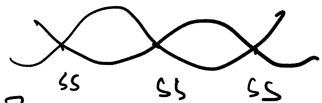
$\mathcal{X}_K$: rigid analytic space$/C$

$\cup|$

$\mathcal{X}_{K,c}$

$M^{\dagger}_{k}(K) :=$ sections of ω^k defined in a strict nbhd of $\mathcal{X}_{K,c}$

$\mathfrak{X}_K / \mathcal{O}_C$



$\{\text{irred. comp}\} \longrightarrow \{\varphi : (\mathbb{Z}/p^n)^2 \twoheadrightarrow \mathbb{Z}/p^n\}$

$C \longmapsto \varphi_C$

$\mathcal{X}_{K,c}$ = tubular nbhd of $\bigcup C^0 \leftarrow$ remove all s.s. pts.

$\varphi_C((1,0)) = 0$

Theorem 2. $k=1$

$$\tilde{H}^1(C)^{la}_1 = \begin{array}{ll} M^{\dagger}_1 & \otimes x_1 \\ \hline H^1(\omega) & \otimes x_1 \\ \hline M^{\dagger}_1 / M_1 & \otimes x_1 \end{array}$$

sub $\to$ $M^{\dagger}_1/M_1 \otimes x_1$

Sen operator

$$\tilde{H}^1(C)^{la,0}_1 = \begin{array}{c} M_1^\dagger / M_1 \\ \hline M_1 \\ \hline H^1(\omega) \\ \hline M_1^\dagger / M_1 \end{array}$$

Sen operator

"Hodge-Tate wt 0 part"

$$\left(\tilde{H}^1(C)^{la}_1\right)^{G_{\mathbb{Q}_p}} \subseteq \tilde{H}^1(C)^{la,0}_1$$

Application I classicality

Hecke algebra.

$$\mathbb{T}(K) \subseteq \mathrm{End}\left(H^1(Y_K(\mathbb{C}), \mathbb{Z}_p)\right)$$

"$\mathbb{Z}_p[T_\ell, S_\ell]$" ℓ good

$$\mathbb{T}(K^p) := \varprojlim_{K_p} \mathbb{T}(K^p K_p)$$

$\circlearrowright$

$$\tilde{H}^1(K^p, \mathbb{Z}_p), \quad \underset{B \circlearrowright}{M_1^\dagger(K^p)} = \varinjlim_n M_1^\dagger(K^p I(p^n))$$

$$N_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \subseteq B,$$

- $U_p \curvearrowright M_1^\dagger(K^p)^{N_0}$, $U_p = \sum_{i=0}^{p-1} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}$

- $\lambda : \mathbb{T}(K^p) \to \overline{\mathbb{Q}}_p$

$\rightsquigarrow \rho_\lambda : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$ s.s.

Theorem 3 If $M_1^\dagger(K^p)[\lambda]^{N_0}$ has a non-zero U_p-eigenvector.

$\rho_\lambda|_{G_{\mathbb{Q}_p}}$ is HT of wts 0, 0

$\Leftrightarrow$ $\rho_\lambda(I_{\mathbb{Q}_p})$ is finite.

then λ is classical, i.e. $M_1(K^p)[\lambda] \neq 0$.

Proof (Sketch)

Assume ρ_λ is irreducible

Suppose λ is NOT classical.

Eichler-Shimura relation

$$\tilde{H}^1(K^p, \mathbb{C})^{la}_{\mu_1}[\lambda \otimes \chi_1] \doteq \rho_\lambda \otimes W$$

$\rho_\lambda|_{G_{\mathbb{Q}_p}}$ is HT $\to$ ||

$$\tilde{H}^1(K^p, \mathbb{C})^{la,0}_{\mu_1}[\lambda] \leftarrow \text{HT wt } 0$$

||

λ is Not classical $\Rightarrow$ "

$$M_1^\dagger(K^p)[\lambda]$$

$$M_1^\dagger(K^p)[\lambda]^{\begin{pmatrix}\mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & \end{pmatrix}} = \rho_\lambda \otimes W^{\begin{pmatrix}\mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1\end{pmatrix}}$$

dim = 2

U_p mult. 1 (q-expansion) $\circlearrowleft$ $\qquad$ $\circlearrowleft$ U_p

Contradiction!

Application II

Theorem 4 $p \geq 3$.

$\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$ cts, irred, odd

- $\rho|_{G_{\mathbb{Q}_\ell}}$ is unramified a.a. ℓ
- $\rho|_{G_{\mathbb{Q}_p}}$ is HT of wts 0, 0.
- mild hypothesis on $\bar{\rho}$.

Then ρ comes from a classical wt 1 eigenform.

Theorem 2.3 + Emerton's local-global compatibility + Colmez's Kirillov model

Main Construction

Scholze: $\mathcal{X}_{K^p} \sim \varprojlim_{K_p} \mathcal{X}_{K^pK_p}$
perfectoid space

$$\pi_{HT} : \mathcal{X}_{K^p} \to \mathcal{F}l = \mathbb{P}^1$$

$$\tilde{H}^1(K^p, C) \cong H^1(\mathcal{X}_{K^p}, \mathcal{O}_{\mathcal{X}_{K^p}})$$

$$= H^1(\mathcal{F}l, \mathcal{O}_{K^p})$$

$GL_2(\mathbb{Q}_p) \curvearrowright \mathcal{O}_{K^p} = \pi_{HT *} \mathcal{O}_{\mathcal{X}_{K^p}}$

Def'n $\mathcal{O}_{K^p}^{la} \subseteq \mathcal{O}_{K^p}$: locally analytic sections

$K_p \curvearrowright U \subseteq \mathcal{F}l$ $\quad \mathcal{O}_{K^p}^{la}(U) := \mathcal{O}_{K^p}(U)^{K_p\text{-}la}$

Fact $\tilde{H}^1(K^p, C)^{la} \cong H^1(\mathcal{F}l, \mathcal{O}_{K^p}^{la})$

$\mathfrak{g} = \mathfrak{gl}_2(C)$

$\mathfrak{g}^0 := \mathcal{O}_{\mathcal{F}\ell} \otimes_C \mathfrak{g} \curvearrowright \mathcal{O}^{la}_{K^P} \supseteq \mathcal{O}_{\mathcal{F}\ell}$

$\cup\!\!|\quad \mathcal{F}\ell \ni x, \quad \mathfrak{n}_x \subseteq \mathfrak{b}_x \subseteq \mathfrak{g}$

$\mathfrak{b}^0 := \{ f \in \mathfrak{g}^0 \mid f_x \in \mathfrak{b}_x \}$

$\cup\!\!|$

$\mathfrak{n}^0 := \{ f \in \mathfrak{g}^0 \mid f_x \in \mathfrak{n}_x \}$

Theorem 5 $\mathfrak{n}^0$ acts trivially on $\mathcal{O}^{la}_{K^P}$ ($\rightsquigarrow$ D-module)

($\mathcal{O}^{la}_{K^P}$ satisfies a first order differential eq'n)

$\mathfrak{b}^0/\mathfrak{n}^0 \curvearrowright \mathcal{O}^{la}_{K^P}$ $\quad\quad \mathfrak{n} \subseteq \mathfrak{b} = \{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \}$

$\wr|$ $\quad\quad \mathfrak{h} = \mathfrak{b}/\mathfrak{n}$

$\mathfrak{h} \hookrightarrow \mathcal{O}_{\mathcal{F}\ell} \otimes \mathfrak{h}$ $\quad\quad \text{"} \{ \begin{pmatrix} * & \\ & * \end{pmatrix} \}$

(horizontal)

$\rightsquigarrow$ an action $\theta_\mathfrak{h}$ of $\mathfrak{h}$ on $\mathcal{O}^{la}_{K^P}$

Harish-Chandra: $\theta_\mathfrak{h}$ encodes infinitesimal character

Theorem 6

$\theta_{\mathfrak{h}}\left(\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$ is a Sen operator on $\tilde{H}^1(K^p, C)^{la}$

$\cup$

$G_{\mathbb{Q}_p}\circlearrowright W$: f.d $/C$

$\circlearrowleft$

θ_{Sen}

$\rightsquigarrow$ gives HT decomposition in Thm 1

Sen operator in Thm 2.

Theorem 7. (Berger-Colmez)

$\tilde{X} = \mathrm{Spa}(B, B^+)$

$\downarrow$ pro-étale G-covering $\qquad$ G: some finite dim'l p-adic Lie group

$X = \mathrm{Spa}(A, A^+)$ smooth $/C$, dim $=1$

$G \curvearrowright B$, $\mathrm{Lie}(G) \curvearrowright B^{la}$

Then some smallness assumption on X.

$\exists \quad \theta \in B \otimes_{\mathbb{Q}_p} \mathrm{Lie}(G) \quad \ni \sum a_i \otimes l_i$

s.t. $\theta(B^{la}) = 0$.

p-adic Cauchy-Riemann operator

<u>Example</u> $\tilde{X} = \mathrm{Spa}(C\langle T^{\pm 1/p^\infty}\rangle, \mathcal{O}_C\langle T^{\pm 1/p^\infty}\rangle)$

$\downarrow \quad G \cong \mathbb{Z}_p$

$X = \mathrm{Spa}(C\langle T^{\pm 1}\rangle, \mathcal{O}_C\langle T^{\pm 1}\rangle)$

$$C\langle T^{\pm 1/p^\infty}\rangle^{la} = \bigcup_n C\langle T^{\pm 1/p^n}\rangle$$

smooth vectors

θ: a generator of $\mathrm{Lie}(G)$

<u>p-adic Simpson correspondence</u>

$\left\{ \begin{array}{l} V: \text{a finite dim'l} \\ \text{rep'n of } G/\mathbb{Q}_p \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{l} \text{Higgs field} \\ \phi_V : B \otimes_{\mathbb{Q}_p} V \to B \otimes V \otimes \Omega^1_{A/C}(-1) \end{array} \right\}$

monoidal in V

Tannakian: $\theta \in B \otimes \mathrm{Lie}(G) \otimes \Omega^1_{A/C}(-1)$

horizontal sections:

$$\ker(\phi_V) \supseteq (B \otimes_{\mathbb{Q}_p} V)^{G\text{-smooth}}$$

$$V = C^{an}(G, \mathbb{Q}_p)$$

$$(B \hat{\otimes}_{\mathbb{Q}_p} V)^G = (B \hat{\otimes} C^{an}(G, \mathbb{Q}_p))^G$$

$$= B^{G-an}$$

$$\theta(B^{G-an}) = 0$$