A categorical Künneth formula for Weil sheaves

(in progress, joint with Tamir Hemo + Jakob Scholbach)

Goal: Sheaf-theoretic version of Drinfeld's Lemma for schemes in characteristic $p > 0$.

§1 Drinfeld's Lemma.

§2 Sheaf-theoretic formulation

§3 Lisse and constructible sheaves.

§4 Weil sheaves.

§5 Overview of proof.

§1 Drinfeld's Lemma.

$\mathbb{F}_q$ finite field of characteristic $p > 0$.

$\overline{\mathbb{F}}/\mathbb{F}_q$ algebraic closure.

$X_1, X_2 / \mathbb{F}_q$ finite type schemes, $X := X_1 \times_{\operatorname{Spec} \mathbb{F}_q} X_2$.

⇝ map on étale fundamental groups

$$(*) \qquad \pi_1(X) \xrightarrow{\overset{?}{\not\cong}} \pi_1(X_1) \times \pi_1(X_2).$$

Example 1) $X_1 = X_2 = \operatorname{Spec} \mathbb{F}_q$ ⇝ $\hat{\mathbb{Z}} \xrightarrow{\Delta} \hat{\mathbb{Z}} \times \hat{\mathbb{Z}}$.

2) Assume $X_1, X_2 / \overline{\mathbb{F}}$, $X_1 = X_2 = \mathbb{A}^1_{\overline{\mathbb{F}}}$. Then

$$U = \{ t^p - t = x_1 \cdot x_2 \} \to \mathbb{A}^2_{\overline{\mathbb{F}}}$$

finite étale degree p. If $\lambda \neq \mu \in \overline{\mathbb{F}}$, then

$$U|_{\{x_1 = \lambda\}} \not\cong U|_{\{x_1 = \mu\}} \text{ over } \mathbb{A}^1_{\overline{\mathbb{F}}}.$$

<u>Definition.</u> The partial Frobenii $\phi_i : X \to X$, $i=1,2$ are the maps

$$\phi_1 = \mathrm{Frob}_{X_1} \times \mathrm{id}_{X_2}, \quad \phi_2 = \mathrm{id}_{X_1} \times \mathrm{Frob}_{X_2}.$$

<u>Remark.</u> $\phi_1 = \phi_2^{-1}$ on $X_{\text{ét}}$.

<u>Theorem.</u> (Drinfeld, Lau, Lafforgue).

(*) induces equivalence

$$\left\{ \begin{array}{c} \text{continuous rep's of} \\ \pi_1(X_1) \times \pi_1(X_2) \text{ on} \\ \text{FinSet} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{continuous rep's } M \text{ of } \pi_1(X) \\ \text{on FinSet together with} \\ \text{commuting } M \cong \phi_i^* M,\ i=1,2 \\ \text{with composition } M \cong \mathrm{Frob}_X^* M \end{array} \right\}$$

<u>Slight change of perspective:</u> Add $X_3 := \mathrm{Spec}\, \bar{\mathbb{F}}$.

$\rightsquigarrow \quad X \times \mathrm{Spec}\, \bar{\mathbb{F}} = X_{1,\bar{\mathbb{F}}} \times_{\mathrm{Spec}\, \bar{\mathbb{F}}} X_{2,\bar{\mathbb{F}}}$ with

$$\phi_{X_i} = \phi_i \times \mathrm{id}_{\mathrm{Spec}\, \bar{\mathbb{F}}}$$

Via $\pi_1(X_{\bar{\mathbb{F}}}) \to \pi_1(X)$ above categories equivalent to

$$\left\{ \begin{array}{c} \text{cont. rep's of } \pi_1(X_{\bar{\mathbb{F}}}) \\ \text{on FinSet with} \\ \text{commuting } M \cong \phi_{X_i}^* M \end{array} \right\}.$$

§2 Sheaf-theoretic formulation.

Fix coefficient ring $\Lambda = \begin{cases} \text{finite}, \ n\cdot\Lambda = 0, \ p \nmid n. \\ E/\mathbb{Q}_\ell \text{ algebraic}, \ \ell \neq p. \\ \mathcal{O}_E \text{ integers} \end{cases}$

For $X/_{\mathbb{F}_q}$ denote.

$$D_{cons}(X^{Weil}, \Lambda) = \left\{ \begin{array}{l} \text{category of constructible} \\ \Lambda\text{-sheaves } M \text{ on } X_{\bar{\mathbb{F}}} \\ \text{together with } M \cong \phi_X^* M \end{array} \right\}$$

and $D_{lis}(X^{Weil}, \Lambda)$ its subcategory of lisse sheaves.

Theorem. The functor $(M_1, M_2) \mapsto M_1 \boxtimes M_2$ induces an equivalence of categories.

$$D_{cons}(X_1^{Weil}, \Lambda) \underset{\Lambda}{\otimes} D_{cons}(X_2^{Weil}, \Lambda) \overset{(**)}{\xrightarrow{\cong}} D_{cons}(X_1^{Weil} \underset{\mathrm{Spec}\,\bar{\mathbb{F}}}{\times} X_2^{Weil}, \Lambda).$$

and similarly for lisse sheaves.

Remarks (1) $\underset{\Lambda}{\otimes}$ induced from Lurie tensor product.

(2) $X^{Weil} = X_{\bar{\mathbb{F}}}/\phi_X$ as prestack over $\mathrm{Spec}\,\bar{\mathbb{F}}$.

(3). Roughly,

(**) fully faithful $\Leftarrow$ Künneth formula for constructible Λ-sheaves.

(**) essentially surjective $\Leftarrow$ Drinfeld's lemma for Λ-coefficients (C. Xue for $\Lambda/\mathbb{Q}_\ell$).

(4). Variant of (*) for

- multiple factors. $X_1, \ldots, X_n$.
- lisse $\mathbb{F}_p$-sheaves
- ind-lisse / ind-constructible sheaves.

§3 Lisse and constructible sheaves.

Classically,

$$D_{cons}(X, \overline{\mathbb{Q}}_\ell) = \operatorname{colim}_{E/\mathbb{Q}_\ell \text{ finite}} \left(\left(\lim_n D_{cons}(X_{\acute{e}t}, \mathcal{O}_E/\mathfrak{m}_E^n) \right) \otimes_{\mathbb{Z}} \mathbb{Q} \right)$$

$\rightsquigarrow$ use pro-étale sheaves + coefficients in condensed rings.

For X scheme, Λ condensed ring,

$$\Lambda_X := p_X^{-1}\Lambda, \qquad p_X : X_{pro\acute{e}t} \to *_{pro\acute{e}t}.$$

Example. $\left\{ \begin{array}{l} \text{compactly generated} \\ \text{T1-topological rings} \end{array} \right\} \xrightarrow[\text{faithful}]{\text{fully}} \{ \text{condensed rings} \}$.

If Λ (associated with) totally disconnected ring, $U \in X_{pro\acute{e}t}$ qcqs, then

$$\Gamma(U, \Lambda_X) = \mathrm{Maps}_{cont}(\pi_0 U, \Lambda) = \mathrm{Maps}_{cont}(U, \Lambda).$$

Denote

(5)

$$D(X, \Lambda) := \left\{ \begin{array}{l} \text{derived category of sheaves} \\ \text{of } \Lambda_X\text{-modules on } X_{proét} \end{array} \right\}.$$

(co-)complete, stable (= homotopy category triangulated), symmetric monoidal $- \otimes_{\Lambda_X} -$, closed $\underline{\mathrm{Hom}}_{\Lambda_X}(-,-)$, $\Gamma(X, \Lambda)$-linear:

$$\mathrm{Mod}_{\Gamma(X,\Lambda)} \to D(X, \Lambda), \quad K \mapsto K_X := \underline{K} \otimes_{\underline{\Gamma(X,\Lambda)}} \Lambda_X.$$

For $f: Y \to X$ of schemes, $f^{-1}\Lambda_X = \Lambda_Y$, so

$$f^* = f^{-1}: D(X, \Lambda) \rightleftarrows D(Y, \Lambda) : f_*$$

<u>Definition</u>. A sheaf $M \in D(X, \Lambda)$ is <u>lisse</u> if it is dualizable for $\otimes_{\Lambda_X}$. It is <u>constructible</u> if for all $U \subset X$ open affine there is finite subdivision $U_i \subset U$ into constructible locally closed such that $M|_{U_i}$ is lisse.

$\rightsquigarrow$ stable, full subcategories

$$D_{lis}(X, \Lambda) \subset D_{cons}(X, \Lambda)$$

Key Lemma. If X w-contractible affine, then equivalence.

$$D_{lis}(X, \Lambda) \xrightarrow{\cong} \mathrm{Perf}_{\Gamma(X,\Lambda)}, \quad M \mapsto R\Gamma(X, M).$$

proof. $R\Gamma(X,-) = \Gamma(X,-)$ monoidal

$\Rightarrow R\Gamma(X,M) \in \mathrm{Mod}^{\text{dualizable}}_{\Gamma(X,\Lambda)} = \mathrm{Perf}_{\Gamma(X,\Lambda)}$.

fully faithful: For lisse sheaves M, N,

$\underline{\mathrm{Hom}}_{\Lambda_X}(M, N) \cong M^\vee \otimes_{\Lambda_X} N \rightsquigarrow$ apply $R\Gamma(X,-)$.

essentially surjective: For $K \in \mathrm{Perf}_{\Gamma(X,\Lambda)}$,

$$R\Gamma(X, K_X) = K. \qquad \square$$

Lemma. The properties "lisse" and "constructible" are local on $X_{\text{proét}}$. The functors of ∞-categories

$$X_{\text{proét}} \ni U \mapsto D_{lis}(U, \Lambda), \ D_{cons}(U, \Lambda)$$

are hypersheaves.

Corollary. A sheaf $M \in D(X, \Lambda)$ is lisse iff for all $U \in X_{\text{proét}}$ w-contractible affine the $\Gamma(U, \Lambda)$-module $R\Gamma(U, M)$ is perfect.

Examples (1) $D_{cons}(*, \Lambda) = D_{lis}(*, \Lambda) = \mathrm{Perf}_{\Lambda_*}$,

$\Lambda_* := \Gamma(*, \Lambda)$ underlying abstract ring.

(2) $X = \hat{\mathbb{Z}}$ profinite set, $\Lambda = \hat{\mathbb{Z}}$ profinite ring.

$$H^0(X, \Lambda) = \mathrm{Maps}_{cont}(\hat{\mathbb{Z}}, \hat{\mathbb{Z}}) \ni f := \mathrm{id}$$

$\rightsquigarrow$ $M := (\Lambda_X \xrightarrow{f} \Lambda_X) \in D_{lis}(X, \Lambda)$ such that

- reduction M mod $m\hat{\mathbb{Z}}$ perfect-locally constant for all $0 \neq m \in \mathbb{Z}$.
- M not perfect-locally constant.

<u>Comparison results.</u>

(1) Λ discrete topological ring.

$$\rightsquigarrow \quad D_{lis}(X, \Lambda) \cong \underbrace{D_{lis}(X_{et}, \Lambda)}_{= \text{ perfect-locally constant on } X_{et}}$$

(2) $\Lambda = \lim \Lambda_i$ sequential limit of condensed rings, surjective transition maps, locally nilpotent kernels

$$\rightsquigarrow D_\bullet(X, \Lambda) \xrightarrow{\cong} \lim D_\bullet(X, \Lambda_i), \quad \bullet \in \{lis, cons\}$$

(3) X qcqs, $\Lambda = \mathrm{colim}\, \Lambda_i$ filtered colimit of cond. rings.

$\rightsquigarrow \operatorname{colim} D_\bullet(X, \Lambda_i) \xrightarrow{\simeq} D_\bullet(X, \Lambda)$, $\bullet \in \{\text{lis}, \text{cons}\}$ (8)

$(4)^{*\text{ tentative}}$ Assume X locally top. Noetherian. Then $M \in D(X, \Lambda)$ lisse iff it is perfect-locally constant on $X_{\text{proét}}$.

§4 Weil sheaves

$X/\mathbb{F}_q$ scheme.

Definition. The <u>Weil pro-étale site</u> $X^{\text{Weil}}_{\text{proét}}$ is the site:

- objects: $U \in (X_{\overline{\mathbb{F}}})_{\text{proét}}$ together with $\varphi: U \to U$ such that
$$\begin{array}{ccc} U & \xrightarrow{\varphi} & U \\ \downarrow & & \downarrow \\ X_{\overline{\mathbb{F}}} & \xrightarrow{\phi_X} & X_{\overline{\mathbb{F}}} \end{array}$$
commutes.
- morphisms: equivariant maps.
- covers: $\{(U_i, \varphi_i) \to (U, \varphi)\}$ such that $\{U_i \to U\}$ cover in $(X_{\overline{\mathbb{F}}})_{\text{proét}}$.

$\rightsquigarrow$ maps of sites

$$(X_{\overline{\mathbb{F}}})_{\text{proét}} \to X^{\text{Weil}}_{\text{proét}} \to X_{\text{proét}}.$$
$$(U_{\overline{\mathbb{F}}}, \phi_U) \hookleftarrow U$$

$$U \longleftarrow (U, \varphi)$$

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<u>Proposition</u>. For condensed ring Λ,

$$(\ast\ast\ast) \quad D(X^{\mathrm{Weil}}, \Lambda) \cong D(X_{\overline{\mathbb{F}}}, \Lambda)^{\phi_X = \mathrm{id}}.$$

<u>Idea</u> $(U, \varphi) := \big(\mathbb{N} \times X_{\overline{\mathbb{F}}}, (n, x) \mapsto (n+1, \phi_X(x)) \big) \in X^{\mathrm{Weil}}_{\mathrm{proét}}$.

$\rightsquigarrow$ equivalence of topoi

$$\mathrm{Shv}(X_{\overline{\mathbb{F}}, \mathrm{proét}}) \cong \mathrm{Shv}\big(X^{\mathrm{Weil}}_{\mathrm{proét}} / (U, \varphi) \big)$$

Use descent for Čech nerve

$$(U, \varphi)_{\bullet / (X_{\overline{\mathbb{F}}}, \phi_X)} \cong (\mathbb{N}^{\bullet} \times X_{\overline{\mathbb{F}}}, \ldots)$$

$\to$ computes homotopy fixed points.

$\square$.

<u>Lemma</u>. Let $X/_{\mathbb{F}_q}$ qcqs, $k/_{\mathbb{F}_q}$ separably closed field, $p: X_k \to X$. Then $A \mapsto p^{-1}(A)$ induces bijection.

$$\left\{ \begin{array}{c} \text{constructible subsets} \\ \text{in } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \phi_X\text{-invariant constructible} \\ \text{subsets in } X_k \end{array} \right\}$$

<u>Definition</u>. A Weil sheaf $M \in D(X^{\mathrm{Weil}}, \Lambda)$ is <u>lisse</u> if it is dualizable. M is <u>constructible</u> if for all open affine $U \subset X$ there is finite subdivision $U_i \subset X$ into constructible locally closed such that

$$M|_{U_i} \in D(U_i^{\mathrm{Weil}}, \Lambda) \text{ is lisse.}$$

Lemma. (∗∗∗) restricts to equivalence

$$\mathcal{D}_\bullet(X^{\mathrm{Weil}}, \Lambda) \cong \mathcal{D}_\bullet(X_{\bar{\mathbb{F}}}, \Lambda)^{\phi_X = \mathrm{id}}, \quad \bullet \in \{\mathrm{lis}, \mathrm{cons}\}.$$

Remark. Λ finite discrete

$$\Longrightarrow \mathcal{D}_\bullet(X^{\mathrm{Weil}}, \Lambda) \cong \mathcal{D}_\bullet(X, \Lambda),$$

but not so for $\Lambda = \mathbb{Z}_\ell, \mathbb{Q}_\ell$.

Remark (Products)

$X_1, \dots, X_n / \mathbb{F}_q \rightsquigarrow$ site $(X_1^{\mathrm{Weil}} \times \dots \times X_n^{\mathrm{Weil}})_{\text{proét}} = \{(U, \varphi_1, \dots, \varphi_n)\}$.

$$\rightsquigarrow \mathcal{D}_\bullet(X_1^{\mathrm{Weil}} \times \dots \times X_n^{\mathrm{Weil}}, \Lambda) \cong \mathcal{D}_\bullet(X_{1,\bar{\mathbb{F}}} \times \dots \times X_{n,\bar{\mathbb{F}}}, \Lambda)^{\phi_{X_1} = \mathrm{id}, \dots, \phi_{X_n} = \mathrm{id}}$$

for $\bullet \in \{\emptyset, \mathrm{lis}, \mathrm{cons}\}$

objects M together with commuting equivalences $M \cong \phi_{X_i}^* M$, $i = 1, \dots, n$.

For $\bullet = \mathrm{cons}$ use:

Proposition Let $X_1, \dots, X_n / \mathbb{F}_q$ qcqs. Then any partial Frobenius invariant constructible closed subset

$$Z \subset X_1 \times \dots \times X_n$$

is a finite union of subsets of the form $Z_1 \times \dots \times Z_n$ for $Z_i \subset X_i$ constructible closed.

§5 Proof of categorical Künneth formula.

X_1, X_2 finite type $/\mathbb{F}_q$, $\Lambda = \begin{cases} \text{finite}, \ n\cdot\Lambda = 0, \ p \nmid n. \\ E/\mathbb{Q}_\ell, \ \ell \neq p. \\ \mathcal{O}_E \end{cases}$

Consider.

$$D_\bullet(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} D_\bullet(X_2^{\mathrm{Weil}}, \Lambda) \xrightarrow{(**)} D_\bullet(X_1^{\mathrm{Weil}} \times X_2^{\mathrm{Weil}}, \Lambda)$$

<u>claim</u>. $(**)$ equivalence for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$.

Mayer-Vietoris arguments $\rightsquigarrow$ reduce to $\bullet = \mathrm{lis}$.

<u>Full faithfulness</u>: For $M_i, N_i \in D_{\mathrm{lis}}(X_i^{\mathrm{Weil}})$,

$$\begin{array}{c} R\mathrm{Hom}_{D(X_1^{\mathrm{Weil}})}(M_1, N_1) \otimes_{\Lambda_*} R\mathrm{Hom}_{D(X_2^{\mathrm{Weil}})}(M_2, N_2) \\ \Big\downarrow \overset{!}{\simeq} \\ R\mathrm{Hom}_{D(X_1^{\mathrm{Weil}} \times X_2^{\mathrm{Weil}})}(M_1 \boxtimes M_2, N_1 \boxtimes N_2) \end{array}$$

M_i lisse ($=$ dualizable) $\rightsquigarrow$ WLOG $M_i = \Lambda_X$.

Have exact triangle

$$R\Gamma(X_i^{\mathrm{Weil}}, N_i) \to R\Gamma(X_{i,\bar{\mathbb{F}}}, N_i) \xrightarrow{\mathrm{id} - \phi^*_{X_i}} R\Gamma(X_{i,\bar{\mathbb{F}}}, N_i)$$

$\rightsquigarrow$ use Künneth formula for $X_{1,\bar{\mathbb{F}}} \times X_{2,\bar{\mathbb{F}}}$

Essential surjectivity:

Put $X := X_1 \times_{\mathrm{Spec}\,\mathbb{F}_q} X_2$. Assume (for simplicity)

$X_i / \mathbb{F}_q$ smooth connected, $X(\mathbb{F}_q) \neq \emptyset$ and

Λ field or Dedekind ring.

enough: objects in heart of standard t-structure lie in essential image.

closed under extensions + direct summands.

Lemma.

$$D_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times X_2^{\mathrm{Weil}}, \Lambda)^{\heartsuit} \simeq \underbrace{\mathrm{Rep}_{\Lambda}(\mathrm{FWeil}(X))}$$

= continuous rep'n on finitely presented Λ-modules.

and $\mathrm{FWeil}(X) := \pi_1(X_{\bar{\mathbb{F}}}) \rtimes \langle \phi_{X_1}^{*\mathbb{Z}}, \phi_{X_2}^{*\mathbb{Z}} \rangle$.

Put $\mathrm{Weil}(X_i) := \pi_1(X_{i,\bar{\mathbb{F}}}) \rtimes \langle \phi_{X_i}^{*\mathbb{Z}} \rangle$.

$\rightsquigarrow$ map of locally profinite groups

$$(*) \quad \mathrm{FWeil}(X) \to \mathrm{Weil}(X_1) \times \mathrm{Weil}(X_2).$$

Drinfeld's lemma for Weil groups.

(*) induces

$$\mathrm{Rep}_\Lambda(\mathrm{Weil}(X_1)\times \mathrm{Weil}(X_2)) \cong \mathrm{Rep}_\Lambda(\mathrm{FWeil}(X))$$

$\rightsquigarrow$ need decomposition results for rep's of product groups

$$W = W_1 \times W_2.$$

<u>Proposition.</u> Let $M \in \mathrm{Rep}_\Lambda(W)$.

(1) Λ algebraically closed field, M simple

$\Rightarrow \exists\, M_i \in \mathrm{Rep}_\Lambda(W_i)$: $M \cong M_1 \boxtimes M_2$.

(2) Λ perfect field, M simple.

$\Rightarrow$ M direct summand of some $M_1 \boxtimes M_2$

(3) Λ Dedekind ring, $\mathrm{Frac}(\Lambda)$ perfect,

$M \in \mathrm{Rep}_\Lambda(W)$ finite projective, $M \otimes \mathrm{Frac}(\Lambda)$ simple

$\Rightarrow$ $\exists$ exact sequence.

$$0 \to M \oplus N \to M_1 \boxtimes M_2 \to \underbrace{T}_{\Lambda\text{-torsion}} \to 0.$$

$\rightsquigarrow$ use dévissage to conclude.