1/

Prismatic Crystals and Crystalline Galois Representations

(jt w/ Bhargav Bhatt)

Let K p-adic field, i.e. complete discretely valued field of mixed characteristic, perfect residue field $k \twoheadleftarrow \mathcal{O}_K \subset K$.

$$\begin{tikzcd} k & \mathcal{O}_K \arrow[l, two heads] \arrow[r, phantom, "\subset"] & K \\ & \pi \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & \end{tikzcd}$$

Thm (B-S) Prismatic F-crystals on $\mathcal{O}_K$ are equivalent to lattices in crystalline G_K-repr's.

$G_K = \mathrm{Gal}(\overline{K}|K)$.

Three Points of View on Prismatic Crystals

Recall. If $(T, \mathcal{O})$ any ringed topos, have fully faithful functor

$$\begin{tikzcd} \{\text{fin. proj. } \mathcal{O}(T)\text{-modules}\} \arrow[d, hook] & M \arrow[d, mapsto] \\ \{\mathcal{O}\text{-modules on } T\}. & M \otimes_{\mathcal{O}(T)} \mathcal{O}. \end{tikzcd}$$

A crystal on $(T, \mathcal{O})$ is a sheaf of $\mathcal{O}$-modules that is locally of this form.

If descent of fin. proj. modules is effective, this is equiv. to the datum of fin. proj. $\mathcal{O}(X)$-module $M(X)$ $\forall X \in T_0 \subseteq T$ + isom. $M(X) \otimes_{\mathcal{O}(X)} \mathcal{O}(Y) \xrightarrow{\sim} M(Y)$ $Y \to X$. with comp. compositi

1). Site $(X_\Delta, \mathcal{O}_\Delta)$.

Recall: A *prism* is a pair

$(\underset{\underset{\phi}{\circlearrowleft}}{A}, I)$, where

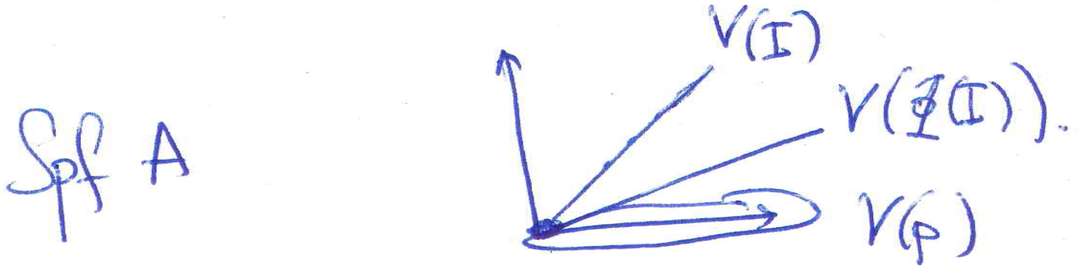
– A p-adically complete ring ~~that~~ with δ-ring structure

$$\delta : A \longrightarrow A$$

in particular

$\phi(x) = x^p + p\,\delta(x) : A \longrightarrow A$ is a ring homom. lifting Frobenius on A/p.

– $I \subseteq A$ Cartier divisor s.th. A I-adically complete $+\ p \in I + \phi(I)$.



for p-adic formal scheme X,

$$X_{\Delta} : \left\{ \begin{array}{l} (A, I) \text{ prism} \\ \quad + \text{ map } \mathrm{Spf}(A/I) \longrightarrow X \end{array} \right\}^{op}.$$

endow with

- $\mathcal{O}_{\Delta} : (A, I) \longmapsto A.$ $\circlearrowleft \phi$
  $\circlearrowleft \phi \supseteq \mathcal{I}_{\Delta} : (A, I) \longmapsto I.$

- topology of (p, I)-completely f. flat maps
  $(A, I) \longrightarrow (B, J).$

$\rightsquigarrow$ crystals on this.

<u>Example</u> $X = \mathrm{Spf}(\mathcal{O}_K)$

Breuil-Kisin prism: $(\mathfrak{S}, (E(u)))$ prism.
$\mathrm{Spf}(\mathfrak{S}/E(u)) = \mathrm{Spf}(\mathcal{O}_K) \Rightarrow X.$

$$\begin{array}{ccc} \mathfrak{S} = W(k)[\![u]\!] & \twoheadrightarrow & \mathcal{O}_K \\ u & \longmapsto & \overset{\in}{\pi} \end{array}$$

$\circlearrowleft$
$\phi : u \mapsto u^p$ kernel $= I = (E(u))$
$\ulcorner$ Eisenstein polynomial.

5/ In fact,

$$\left(\mathfrak{S}, (E(u))\right) \longrightarrow * \text{ . covers}$$

final object of X_{Δ}.

⇒ Crystals on X_{Δ} are equiv. to fin. proj. $\mathfrak{S}$-modules $\mathfrak{M}$

+ isom.

$$\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{(2)} \cong \mathfrak{S}^{(2)} \otimes_{\mathfrak{S}} \mathfrak{M}$$

satisfying cocycle condition,

where

$$\mathfrak{S}^{(2)} = W(k)[\![u_1, u_2]\!]\left\{\overset{\overset{x}{\shortparallel}}{\frac{u_1-u_2}{E(u_1)}}\right\}$$

adjoin as δ-ring, and (p, I)-complete.

$$\left(\left(\mathfrak{S}^{(2)}, (E(u_1))\right) = (\mathfrak{S}, E(u)) \times_{X_{\Delta}} (\mathfrak{S}, E(u)) \right)$$

<u>Example.</u> $K = \mathbb{Q}_p$, $\pi = p$.

$E(u) = u - p$. $x = \frac{u_1-u_2}{u_1-p}$ $\delta(x) = \dfrac{\frac{u_1^p-u_2^p}{u_1^p-p} - \left(\frac{u_1-p}{u_2-p}\right)^p}{p}$

2) ~~Set~~ Crystals on (X_{qsyn}, Δ_{-}).

Assume X quasisyntomic.
(e.g. Noetherian + local complete intersection)

site X_{qsyn}: $\mathrm{Spf}\, A \longrightarrow X$.
quasisyntomic.
(p-completely flat + cotangent complex p-complete Tor amplitude in $[0,1]$.)
Covers: qsyntomic covers.

locally on X_{qsyn},
A is quasiregular semiperfectoid.

weakening of being quotient by regular sequence. (→ quasiregular)

quotient of perfectoid. (→ semiperfectoid)

For such A, $(\mathrm{Spf}\, A)_{\Delta}$ has final object (Δ_A, I)

$$\circlearrowleft \phi$$

$$A \longrightarrow \Delta_A / I.$$

use $A \longmapsto \Delta_A$ as structure sheaf on X_{qsyn}.

7/ Example $X = \mathrm{Spf}(\mathcal{O}_K)$.

$C = \hat{K}$ then $\mathcal{O}_C$ perfectoid

$X' = \mathrm{Spf}\,\mathcal{O}_C \longrightarrow X = \mathrm{Spf}\,\mathcal{O}_K$ q-syn cover.

$$\begin{tikzcd}
I \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & \Delta_{\mathcal{O}_C} \arrow[r, equal] \arrow[dr] & A_{\inf}(\mathcal{O}_C) \arrow[r, equal] \arrow[d, two heads, "\theta"] & W(\mathcal{O}_C^\flat) \arrow[loop right, "\varphi"] & \\
(d). & & \mathcal{O}_C. & {[\pi^\flat]} \arrow[u, phantom, "\in"] & u \in \mathfrak{S} \arrow[l, maps to]
\end{tikzcd}$$

$d = p - [p^\flat]$ $\qquad p^\flat \in \mathcal{O}_C^\flat$

$$p^\flat = (p, p^{1/p}, p^{1/p^2}, \dots).$$

$$X' \times_X X' = \mathrm{Spf}\,(\mathcal{O}_C \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C).$$

$$\Delta_{\mathcal{O}_C \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C} = \left(A_{\inf} \hat{\otimes}_{W(k)} A_{\inf}\right)\left\{\frac{[\pi_1^\flat]-[\pi_2^\flat]}{d}\right\}$$

As nasty as before.

crystals $\cong$ fin. proj. $\underset{\substack{\shortparallel \\ A_{\inf}}}{\Delta_{\mathcal{O}_C}}$-modules M

+ isom. $M \otimes_{\Delta_{\mathcal{O}_C}} \Delta_{\mathcal{O}_C \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C} \cong \Delta_{\mathcal{O}_C \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C} \otimes_{\Delta_{\mathcal{O}_C}} M$

+ cocycle condition.

3) Vector bundles on X^{Δ}.

Observation:

$\mathfrak{S}^{(2)}$ is (p,I)-completely flat$/\mathfrak{S}$,

$$\mathfrak{S}^{(3)} = \mathfrak{S}^{(2)} \hat{\otimes}_{\mathfrak{S}} \mathfrak{S}^{(2)}.$$

$\rightsquigarrow$ $\operatorname{Spf} \mathfrak{S}^{(2)} \rightrightarrows \operatorname{Spf} \mathfrak{S}$

induces a ((p,I)-compl.) flat equivalence relation on $\operatorname{Spf} \mathfrak{S}$.

$$\Rightarrow \quad \overset{\phi}{\circlearrowright} X^{\Delta} = \operatorname{Spf} \mathfrak{S} / \operatorname{Spf} \mathfrak{S}^{(2)}$$

is formal stack,

Vector bundles on X^{Δ} = prismatic crystals.

X^{Δ} independent of choices.

Thm (Bhatt-Lurie, Drinfeld).

$$X^{\Delta}(A) = \left\{ \begin{array}{l} \underset{}{\overset{p \in I+\phi(I)}{\text{primitive}}} \text{ ideals } I \subseteq \overset{\phi}{\circlearrowright} W(A) \\ + \operatorname{Spf}(W(A)/I) \longrightarrow X \end{array} \right\}.$$

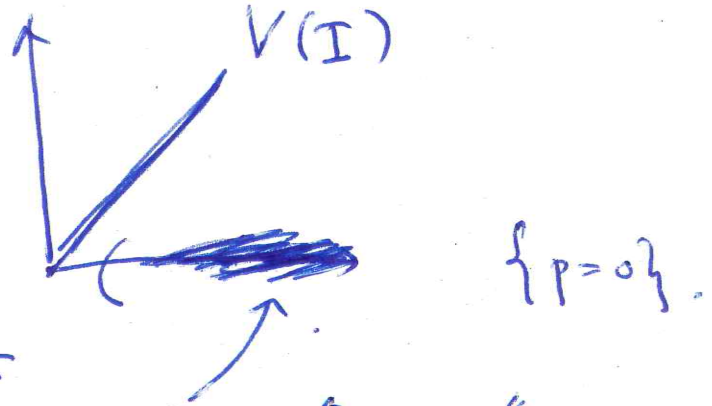
Definition. A prismatic F-crystal on X is a prismatic crystal $\mathcal{M}$ on X + isom.

$$F : (\phi^* \mathcal{M})\left[\frac{1}{I}\right] \cong \mathcal{M}\left[\frac{1}{I}\right].$$

similar notions if $\mathcal{O}_{\Delta}$ is replaced by $\mathcal{O}_{\Delta}\left[\frac{1}{p}\right]$,

$\mathcal{O}_{\Delta}[1/I]^{\wedge}_{p}$, ...

A) Relation to Galois Rep's.



for $\mathcal{O}_C$, get

$$W(C^\flat) = \mathcal{O}_{\Delta}\left[\frac{1}{I}\right]^{\wedge}_{p}.$$

$\phi \circlearrowleft$

"infinitesimal nbhd of "étale point"".

8/ **Thm**. $X = \mathrm{Spf}(\mathcal{O}_K)$.

1). F-crystals for $(X_\Delta, \mathcal{O}_\Delta[\frac{1}{I}]^\wedge_p)$

$\cong \mathrm{Rep}_{\mathbb{Z}_p}(G_K)$.

$9^5/n$ 2). – " – $(X_\Delta, \mathcal{O}_\Delta[\frac{1}{I}]^\wedge_p[\frac{1}{p}])$

lescent

+ étale in prismatic cohom.

comparison. $\cong \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.

3) F-crystals for $(X_\Delta, \mathcal{O}_\Delta)$

$\downarrow$ fully faithful

descent along B-K prism

+ Thm of Kisin. – " – $(X_\Delta, \mathcal{O}_\Delta[\frac{1}{I}]^\wedge_p)$.

idem after $[\frac{1}{p}]$.

4). Beauville-Laszlo gluing + extend VB over a codim 2 point.

$$\begin{tikzcd}
\{\text{F-cryst. for } (X_\Delta, \mathcal{O}_\Delta)\} \arrow[r] \arrow[d, "3)"] & \{\text{– " –}\ (X_\Delta, \mathcal{O}_\Delta[\frac{1}{p}])\} \arrow[d, "3)"] \\
\{\text{– " –}\ (X_\Delta, \mathcal{O}_\Delta[\frac{1}{I}]^\wedge_p)\} \arrow[r] \arrow[d, equal, "1)"] & \{\text{– " –}\} \arrow[d, equal, "2)"] \\
\mathrm{Rep}_{\mathbb{Z}_p}(G_K) & \mathrm{Rep}_{\mathbb{Q}_p}(G_K).
\end{tikzcd}$$

Cartesian.

5). image ~~her~~ $\overset{\text{easy}}{\subseteq}$ crystalline G_K-repr.

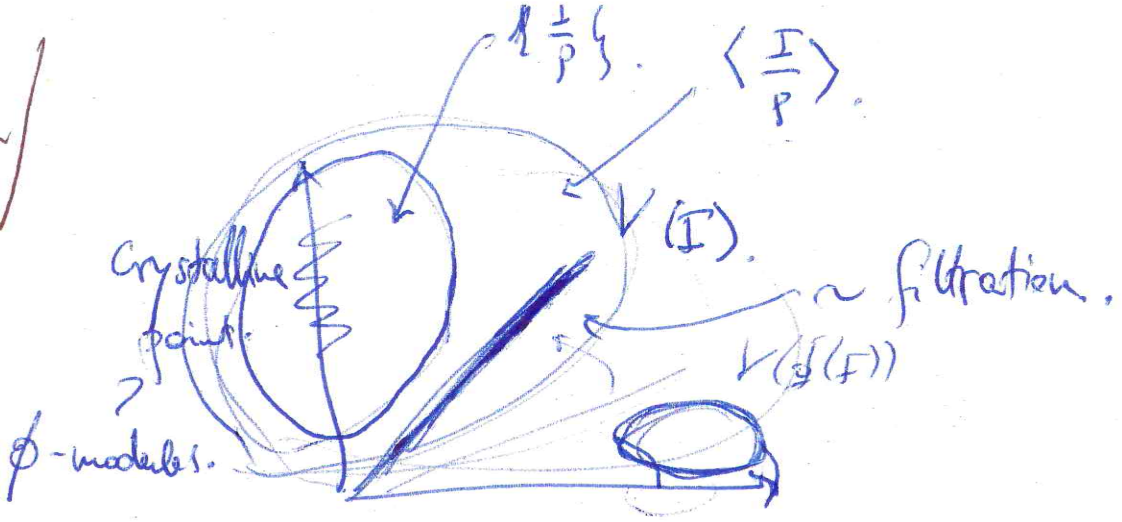
<u>remains:</u> <u>all</u> crystalline G_K-repr. arise from ~~pri~~ ~~Ca~~ F-crystals for $(X_\Delta, \mathcal{O}_\Delta[\frac{1}{p}])$.

B) <u>Relation to filtered ϕ-modules.</u>

<u>Recall</u> <u>Thm</u> (Fontaine, Colmez-Fontaine)

$$\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(G_K) \simeq \left\{ \begin{array}{l} \text{~~fil~~ weakly admissible} \\ \text{filtered } \phi\text{-modules:} \\ \left(\begin{array}{c} N \\ \circlearrowleft \\ F \end{array} \cancel{F}, \mathrm{Fil}^\bullet N_K \right) : \\ \quad \begin{array}{ll} N & \text{f.d. } K_0\text{-v.s.} \\ \circlearrowleft & \\ F & \phi\text{-linear isom.} \end{array} \\ + \quad \mathrm{Fil}^\bullet N_K \subseteq N_K = N \otimes_{K_0} K \end{array} \right\}.$$

$$\begin{array}{c} K_0 = W(k)[\frac{1}{p}] \\ \cap| \\ K. \end{array}$$



to localize near crystalline point on $V(I)$, look at following sheaves on X_{qsyn}.

$$\begin{tikzcd} - & A \arrow[r, mapsto] & \Delta_A\{\frac{I}{p}\} = \Delta_{A/p}, \arrow[loop above, "\phi"] \arrow[d, bend left, "\phi"] \\ - & A \arrow[r, mapsto] & \Delta_A\langle\frac{I}{p}\rangle \arrow[u] \arrow[loop right, "\phi"] \end{tikzcd}$$

- $A \longmapsto \left(\Delta_A[\frac{1}{p}]\right)^{\wedge}_{I}$.
- $A \longmapsto \left(\Delta_A[\frac{1}{p}]\right)^{\wedge}_{I}[\frac{1}{I}]$.

A quasiregular semiperfectoid, p-torsion free.

(e.g. $\mathrm{Spf}\, A \in X_{qsyn}$.)

13/ **Thm** 1). F-crystals for $\Delta_{-}\{\frac{I}{p}\}[\frac{1}{p}]$ $\Big\}$ F-crystals for $(\mathrm{Spf}(\mathcal{O}_K(p)_{\Delta}, \mathcal{O}_{\Delta}[\frac{1}{p}]))$

are equiv. to φ-modules $\&\, \substack{N\\ \varphi\\ F}$.

2).

$$\begin{tikzcd}
 & \text{F-crystals for } \Delta_{-}\langle\frac{I}{p}\rangle[\frac{1}{p}]. \arrow[dl] \arrow[dr] & \\
\text{F-crystals for } \Delta_{-}\{\frac{I}{p}\}[\frac{1}{p}] \arrow[dr, "\varphi^*"'] & \cong \text{filtered } \varphi\text{-modules.} \arrow[u] & \text{cryst. for } \Delta_{-}[\frac{1}{p}]^{\wedge}_{I}. \arrow[dl] \\
 & \text{cryst. for } \Delta_{-}[\frac{1}{p}]^{\wedge}_{I}[\frac{1}{I}]. &
\end{tikzcd}$$

Cartesian.

3). crystals for $\Delta_{-}[\frac{1}{p}]^{\wedge}_{I}$ ~~are equ~~

embed fully faithfully into

log-connections on $K[\![t]\!]$.

$(M \quad K[\![t]\!]\text{-mod.} + \nabla : M \longrightarrow M\frac{dt}{t}\,)$

4) crystals for $\Delta_{-}[\frac{1}{p}]^{\wedge}_{I}[\frac{1}{I}]$.

embed fully faithfully into connection over $K(\!(t)\!)$.

5). F-crystals for $\Delta_{-}[\frac{1}{p}]$.

qsyn descent + similar result for O_C.

$\Big\downarrow$ fully faithful.

F-crystals for $\Delta_{-}\langle\frac{I}{p}\rangle[\frac{1}{p}]$.

$\cong$

filtered ϕ-modules.

6). essential image of functor in 5)

= weakly admissible filtered ϕ-modules.

Key: Step 6).

Use qsyn descent along

$$X' = \mathrm{Spf}\ O_C \to X = \mathrm{Spf}\ O_K.$$

Over O_C, use

Thm (Kedlaya). étale ϕ-modules over Robba ring

$\cong$.

étale ϕ-modules over bounded Robba ring.

+ "weak admissibility".

$\Downarrow$

étale ϕ module.

15/ remains: descent datum, a priori with coefficients in

$$\Delta_{\mathcal{O}_C \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C} \left\langle \frac{I}{p} \right\rangle \left[\frac{1}{p}\right],$$

has coefficients in

$$\Delta_{\mathcal{O}_C \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C} \left[\frac{1}{p}\right].$$

Trick: Have $M / \Delta_{\mathcal{O}_C}$.

automatically have F-equiv. isom.

$$M\left[\frac{1}{q-1}\right] \cong \Delta^n_{\mathcal{O}_C}\left[\frac{1}{q-1}\right]$$

$$q = [\varepsilon^{1/p}]. \qquad \varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots).$$

enough:

Lemma Let $A = \mathcal{O}_C \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C$.

$q_1, q_2 \in \Delta_A$.

$$\begin{tikzcd}
\Delta_A\left[\frac{1}{p(q_1-1)(q_2-1)}\right]^{\phi=1} \arrow[d, "\text{is bijective.}"] \\
\Delta_A\left\langle \frac{I}{p} \right\rangle \left[\frac{1}{p(q_1-1)(q_2-1)}\right]^{\phi=1}
\end{tikzcd}$$

This follows from

Thm (Antieau–Mathew–Morrow–Nikolaus)

"Beilinson fibre sequence".

analogue of fundamental exact seq.

$$0 \to \mathbb{Q}_p \to (B_{crys})^{\phi=1} \to B_{dR}/B_{dR}^{+} \to 0$$

valid for any quasiregular semiperfectoid.