Eva Viehmann, Rampage-Seminar, 17.9.20

Newton strata in the weakly admissible locus

1) Motivation - the admissible locus

p prime

$\mathbb{X}$: p-div group over $\overline{\mathbb{F}}_p$

$N = \mathbb{D}(\mathbb{X})_{\breve{\mathbb{Q}}_p}$ the covar. Dieud. crystal.

$\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{un}}$ $\qquad \dim N = ht\, \mathbb{X}$

$K/\breve{\mathbb{Q}}_p$, $\mathcal{O}_K$

p-adic period map

$$\pi : \left\{ (X,\rho) \;\middle|\; \begin{array}{l} X \text{ a } p\text{-div gp.} /\mathcal{O}_K \\ \rho : X \otimes \mathcal{O}_K/p \longrightarrow \mathbb{X} \otimes \mathcal{O}_K/p \text{ a QI} \end{array} \right\} \longrightarrow Gr_c(N)(K)$$

$c = \operatorname{codim} \mathbb{X}$

$$(X,\rho) \longmapsto \text{Hodge filtr.: } \mathbb{D}(X)_K \xrightarrow{\sim} N \otimes K$$

$$\mathrm{Lie}(X^\vee)^\vee \otimes K \hookrightarrow \mathbb{D}(X)_K$$

<u>Qu.</u>: What is the image?

More generally: G: reductive gp $/\mathbb{Q}_p$

$b \in G(\breve{\mathbb{Q}}_p)$, μ a minuscule cochar. of G

s.t. $[b] \in B(G,\mu)$

$\{g^{-1} b \sigma(g) \mid g \in G(\breve{\mathbb{Q}}_p)\}$

$\mathcal{F}(G,\mu)$ ass. flag variety

(G q' split $\rightsquigarrow$ P_μ : the parabolic def. by μ,
$\mathcal{F}(G,\mu) = G/P_\mu$)

p-adic period map (Rapoport–Zink, Scholze)

$$\pi : \breve{\mathcal{M}}(G,b,\mu)_{K \subset G(\mathbb{Q}_p)} \longrightarrow \mathcal{F}(G,\mu)$$

local Shim. var. étale morph. of rig. sp.

$\operatorname{im} \pi = \mathcal{F}(G,\mu)^a$: admissible locus, open in $\mathcal{F}(G,\mu)$

Ex.: $G = D_{1/n}^\times$ b basic, $\mu = (1,0,\dots,0)$

inner form of GL_n corr. to $[b^{-1}]$

$$\mathcal{F}(G,\mu)^a = \Omega = \mathbb{P}^{n-1} \setminus \underset{\substack{H:\ \mathbb{Q}_p\text{-rat.}\\ \text{hyperplane}}}{\bigcup H} \subseteq \mathbb{P}^{n-1} = \mathcal{F}(G,\mu)$$

(2) G-bundles on the FF curve and Newton strata

(Fargues, Caraiani–Scholze)

$C/\mathbb{Q}_p$ alg. closed complete, $C^\flat$ its tilt

$\rightsquigarrow$ have Fargues–Fontaine curve $X/\mathbb{Q}_p$ (for $C^\flat$)

a 1-dim Noeth. regular scheme $/\mathbb{Q}_p$

and $\infty \in X$ with $k(\infty) = C$

$$\hat{\mathcal{O}}_{X,\infty} = B_{dR}^+(C)$$

G a reductive group over $\mathbb{Q}_p$

(for this talk: G q' split)

A G-bundle on X is a G-torsor on X loc. triv. for étale topology

<u>Thm</u> (Fargues) Have a bijection of pointed sets

$$B(G) \xrightarrow{\cong} \{G\text{- bd. on } X\}/_{\cong}$$
$$[b] \longleftrightarrow \mathcal{E}_b$$

$$B(G) = \{[b] \mid b \in G(\breve{\mathbb{Q}}_p)\}$$

Classification:

$[b] \rightsquigarrow$ (i) Kottwitz ~~pt~~. $\kappa_G(b) \in \pi_1(G)_\Gamma$ Galois coinv.

$[GL_n : v(\det b)]$

(ii) Newton point: $\nu_b \in X_*(A)_{\mathbb{Q}, \mathrm{dom}}$

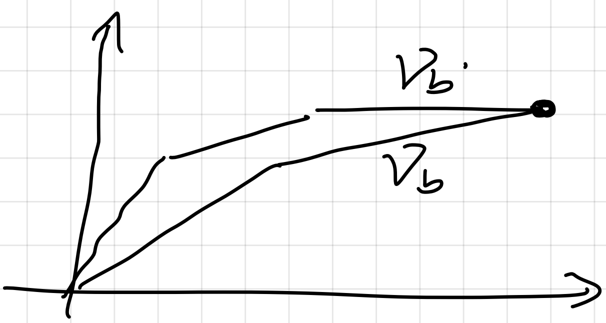
Choose: A max. unram. torus $\subseteq T = C_G(A) \subseteq B$ Borel

$[GL_n :$ $(\breve{\mathbb{Q}}_p^n, b\sigma)$ isocrystal $\rightsquigarrow$ Newton pol. $\in \mathbb{Q}^n_{\mathrm{dom}}]$

partial order: $[b] \leq [b']$

$\iff \kappa_G(b) = \kappa_G(b')$

$\nu_{b'} - \nu_b$ non-neg. $\mathbb{Q}$-lin comb. of pos. coroots



fix $[b] \in B(G)$ basic i.e. ν_b central

let $x \in G(B_{dR})$ $\quad (B_{dR} = B_{dR}(C))$

glue

$\mathcal{E}_b|_{X \setminus \{\infty\}}$ and trivial bdl. $\mathcal{E}_1$ on $\mathrm{Spec}(\underbrace{B_{dR}^+}_{\hat{\mathcal{O}}_{X,\infty}})$

(Beauville-Laszlo) via x

$\Rightarrow$ obtain a G-bundle $\mathcal{E}_{b,x}$ on X

<u>Qu.</u>: $\mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}$, $[b'] \in B(G)$ what is b'?

<u>Remarks</u>: 1) only depends on

$$x \in Gr_G^{B_{dR}}(C) = G(B_{dR})/G(B_{dR}^+)$$

2) $Gr_G^{B_{dR}}(C) \underset{\text{Cartan}}{=} \coprod_{\mu \in X_*(A)_{dom}} \underbrace{G(B_{dR}^+)\mu(t)^{-1}G(B_{dR}^+)/G(B_{dR}^+)}_{Gr_{G,\mu}^{B_{dR}}(C) \ni x}$

Caraiani-Scholze, Rapoport: Let $[b'] \in B(G)$ then there is $x \in Gr_{G,\mu}^{B_{dR}}(C)$ s.t. $\mathcal{E}_{b,x} \cong \mathcal{E}_{b'}$

$\iff [b'] \in B(G, \mu, b) = \{ [b'] \mid$

(i) $\kappa_G(b') = \kappa_G(b) - \mu^{\#}$ $\leftarrow$ image of μ in $\pi_1(G)_\Gamma$

(ii) $\nu_{b'} \leq \nu_b (\mu^{-1})_{dom}^{\diamond}$ $\leftarrow$ galois average $\}$

3) Get decomposition

$$\mathrm{Gr}_{G,\mu}^{B_{dR}} = \coprod_{[b']\in B(G,\mu,b)} \underbrace{\mathrm{Gr}_{G,\mu}^{B_{dR},[b']}}$$

$\rightsquigarrow x \in \mathrm{Gr}(C)$ s.t. $\mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}$

into locally closed gener. subsets

"Newton strata"

4) μ minuscule, $b \in B(G,\mu)$, $[1] \in B(G,\mu,b)$ unique basic el.

$\Rightarrow \mathrm{Gr}_{G,\mu}^{B_{dR},1}$ open, coincides with admissible locus

5) Thm 1 (V.) Hansen (b basic) $\overline{\mathrm{Gr}_{G,\mu}^{B_{dR},[b']}} = \bigcup_{\substack{[b'']\geq[b'] \\ [b'']\in B(G,\mu,b)}} \mathrm{Gr}_{G,\mu}^{B_{dR},[b'']}$

Idea: - Hansen: LHS is a union of Newton strata for $[b'']$ with $\{\mathcal{E}_{b''}\} \in \overline{\{\mathcal{E}_{b'}\}}$ in Bun_G

- V. determine topology on Bun_G.

3 The weakly admissible locus

Let P be a parabolic sgp. of G

$\mathcal{E}$ a G-bundle on X

Def.: A reduction of $\mathcal{E}$ to P is a P-bundle $\mathcal{E}_P$ s.t. $\mathcal{E}_P \times_P G \cong \mathcal{E}$

Have a bij. $(x \in Gr_G^{B_{dR}}(C))$

$\{$reductions of $\mathcal{E}$ to $P\} \longleftrightarrow \{$red. of $\mathcal{E}_x$ to $P\}$

Def.: Let $[b] \in B(G)$ basic, $\mu \in X_*(A)_{dom}$
Then $x \in Gr_{G,\mu}^{B_{dR}}(C)$ is weakly adm. $\Longleftrightarrow$
For every std. par. $P \subseteq G$ with std. Levi M and every reduction b_M of $[b]$ to M ($b_M = g b \sigma(g^{-1}) \in M(\breve{\mathbb{Q}}_p)$)
and every $\chi \in X^*(P/Z_G)_{dom}$ we have

$$\deg \chi_*((\mathcal{E}_{b,x})_P) \leq 0$$

$\mathcal{E}_{b_M}^M \times_M P =: (\mathcal{E}_b)_P$
$\rightsquigarrow (\mathcal{E}_{b,x})_P$

Remarks: 1)
Have Bialynicki-Birula map

$$Gr_{G,\mu}^{B_{dR}} = G(B_{dR}^+)\mu(t^{-1})G(B_{dR}^+)/G(B_{dR}^+) \longrightarrow \underbrace{\mathcal{F}(G,\mu)}_{G/P_\mu}$$

▷ for μ minuscule, this is an isom.
On $\mathcal{F}(G,\mu)$ have weakly adm. locus $\mathcal{F}(G,\mu,b)^{wa}$ defined by Rapoport-Zink

μ minuscule : $Gr_{G,\mu}^{B_{dR},wa} \xrightarrow[BB]{\cong} \mathcal{F}(G,\mu,b)^{wa}$
(Chen-Fargues-Shen)

2) Complement of $Gr_{G,\mu}^{B_{dR},wa}$ is a profinite union of translates (under $J_b(\mathbb{Q}_p)$) of certain U-orbits
U : unip. radical of B

3) Let $Gr_{G,\mu}^{B_{dR}, b_0'}$ for

$b_0' \in B(G, \mu, b)$ the unique basic element

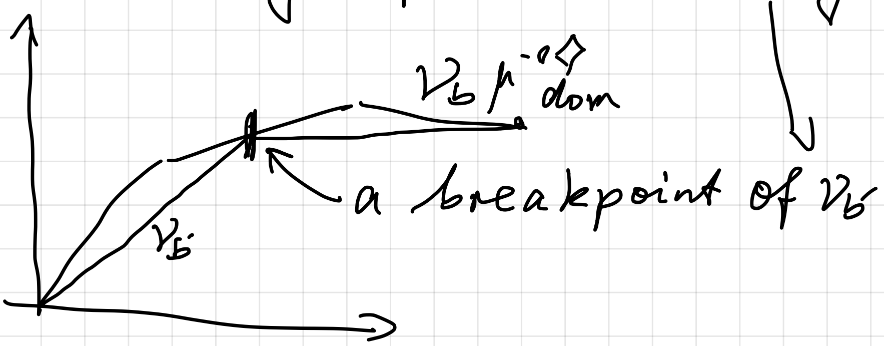
$\subseteq Gr_{G,\mu}^{B_{dR}, wa}$ ("adm. $\Rightarrow$ weakly adm.")

! Coincide only in exceptional cases (CFS, Shen)

(4) Which other Newton strata intersect $Gr_{G,\mu}^{B_{dR}, wa}$?

$[b]$ basic, $\mu \in X_*(A)_{dom}$

Def.: $[b'] \in B(G, \mu, b)$ is Hodge-Newton decomposable if $\nu_{b'} \doteq \nu_b (\mu^{-1})^{\diamond}_{dom}$ in $\pi_1(M)_\Gamma$ for some ~~proper~~ std. Levi subgroup M containing the centralizer of $\nu_{b'}$



Thm 2 (V.) Let $[b'] \in B(G, \mu, b)$. Then

$$Gr_{G,\mu}^{B_{dR}, b'} \cap Gr_{G,\mu}^{B_{dR}, wa} \neq \emptyset \Rightarrow [b'] \text{ is HN-indecomposable}$$

Previously known: (μ minuscule)

(1) Hartl: particular cases for GL_n

(2) "$\Rightarrow$" Chen-Fargues-Shen

(3) Existence of some non-basic b' for which non-emptiness holds

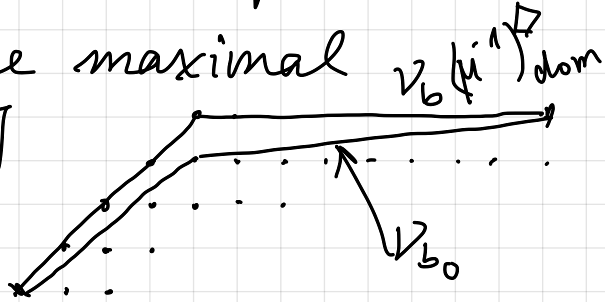
(4) Chen: part. cases for GL_n, b' "close to the basic el"

(5) Chen: Conjectured thm.

(6) Shen: (3) for non-minuscule μ

Idea: (0) "$\Rightarrow$" use a Hodge-Newton decomposition

"$\Leftarrow$" (1) $B(G,\mu,b)$ has a unique maximal HN-indecomp. element $[b_0]$



Have (Thm 1) $\overline{Gr_{G,\mu}^{B_{dR},b'}} \supseteq Gr_{G,\mu}^{B_{dR},b_0}$

b' HN-indec

$Gr_{G,\mu}^{B_{dR},wa}$ is open $\Longrightarrow$ enough to consider $[b'] = [b_0]$

(2) $\dim Gr_{G,\mu}^{B_{dR},b_0} = \langle 2\rho, \nu_b(\mu^{-1})^\diamond_{dom} - \nu_{b_0} \rangle$ (Caraiani-Scholze, Fargues-Scholze)

Recall: $x \in Gr_{...}(C)$ not w.a. $\Longleftrightarrow$

there is a std par. P, M, reduction b_M of $[b]$

and $\chi \in X^*(P/Z_G)_{dom}$ s.t.

$$\deg \chi_*((\mathcal{E}_{b,x})_P) > 0 \qquad (*)$$

(3) Maximality of b_0 $\Longrightarrow$ $(\mathcal{E}_{b,x})_P \times_P M$ is a red. of $\mathcal{E}_{b,x}$ to M

(4) Dimension calculation for the locus of $x \in Gr_{G,\mu}^{B_{dR},b'}(C)$ not w.a.

with $(*)$ for a given datum $P, M, b_M, \deg$, isom. class of $(\mathcal{E}_{b,x})_P \times_P M$

get: Dimension $<$ dim. of Newton stratum.