Perfectoid covers of abelian varieties and the weight-monodromy conjecture

Peter Wear

University of Utah

October 8, 2020

Outline

- Background on the weight-monodromy conjecture
- Overview of Scholze's strategy, using perfectoid covers of toric varieties
- Moving this strategy to abelian varieties

Motivation: L-functions of varieties

For $X/\mathbb{Q}$ a proper smooth variety of dimension d, to any integer $i \in [0, 2d]$ we associate an L-function

$$L_{X,i}(s) = \prod_v L_{X_v,i}(s).$$

This is a product of local factors over places v of $\mathbb{Q}$. It is expected to satisfy a functional equation and analogue of the Riemann hypothesis. It should encode lots of global information about X.

Conjecture (Birch and Swinnerton-Dyer (1965))

The L-function associated to an elliptic curve has a zero at 1 *precisely when the elliptic curve has infinitely many rational points.*

Motivation: local factors at primes of bad reduction

When X has good reduction at a prime p, the local factor at p is relatively well-understood thanks to the Weil conjectures. The Riemann hypothesis implies that the poles of the ith local factor have real part $i/2$.

To get global information, we need to understand the bad reduction picture as well.

Weight-monodromy implies that at primes of bad reduction, the poles of the ith local factor have real part at most $i/2$.

The weight-monodromy conjecture: setup

$p \neq \ell$ primes, $X/\mathbb{Q}_p$ a proper, smooth variety

Finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space $V := H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)$.

Galois representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(V)$.

$$1 \to I_{\mathbb{Q}_p} \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to 1.$$

Fix $\Phi \in G_{\mathbb{Q}_p}$ mapping to geometric Frobenius

Definition

The ith local factor of X at p is $\det(1 - p^{-s}\Phi | V^{I_{\mathbb{Q}_p}})^{-1}$.

The weight-monodromy conjecture: weights

The weight filtration on V comes from the eigenspaces of Φ.

Rapoport-Zink, de Jong: we have

$$V = \oplus_{k=0}^{2i} W'_k$$

where the eigenvalues of Φ acting on W'_k are Weil numbers of weight k.

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Different lifts Φ give different decompositions of V, but the same *weight filtration*

$$W_0 \subset W_1 \subset \cdots \subset W_{2i} = V$$

where $W_j := \oplus_{k=0}^{j} W'_k$.

The weight-monodromy conjecture: monodromy

$$1 \to I_{\mathbb{Q}_p} \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to 1$$

The monodromy filtration on V comes from the pro-ℓ part of the tame inertia.

Grothendieck: The action of the pro-ℓ part of $I_{\mathbb{Q}_p}$ on V leads to the *monodromy operator* $N : V \to V$. It is nilpotent, and gives rise to the monodromy filtration $\{V_j\}$ of V.

This filtration satisfies $N(V_j) \subset V_{j-2}$, and $N^j(gr^N_{i+j} V) \cong gr^N_{i-j} V$.

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For any lift Φ, $N\Phi = p\Phi N$, so $NW_j \subset W_{j-2}$.

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Conjecture (Deligne (1971))

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Deligne proved this over $\mathbb{F}_p((t))$ (1980)

In mixed characteristic, known for dimension at most 2 by Rapoport-Zink (1982) $+$ de Jong's alterations. Various other special cases are known.

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Theorem (W.)

The weight-monodromy conjecture holds for complete intersections in abelian varieties.

Scholze's approach: Perfectoid fields

$$K := \mathbb{Q}_p(p^{1/p^\infty})^\wedge = \Big(\bigcup_{n\in\mathbb{N}} \mathbb{Q}_p(p^{1/n})\Big)^\wedge \text{ is a perfectoid field.}$$

$$K^\flat := \varprojlim_{x\mapsto x^p} K \cong \mathbb{F}_p((t^{1/p^\infty}))^\wedge \text{ is the tilt.}$$

$$\sharp : K^\flat \to K \text{ projects onto the initial term}$$

Theorem (Fontaine-Wintenberger (1979))

The absolute Galois groups G_K and $G_{K^\flat}$ are isomorphic.

Can we transfer Deligne's result from $G_{K^\flat}$ to G_K?

Perfectoid spaces

Perfectoid spaces geometrize this isomorphism.

$$\mathbb{P}_K^{n,\mathrm{perf}}$$

$$\begin{tikzcd}
\vdots \arrow[d, "\varphi"] & \\
\mathbb{P}_K^{n,\mathrm{ad}} \arrow[d, "\varphi"] & (x_0 : \cdots : x_n) \arrow[d, mapsto] \\
\mathbb{P}_K^{n,\mathrm{ad}} & (x_0^p : \cdots : x_n^p)
\end{tikzcd}$$

Behaves like an inverse limit on topological spaces and étale topoi.

Perfectoid spaces

Perfectoid spaces geometrize this isomorphism.

$$\begin{tikzcd}
\mathbb{P}^{n,\mathrm{perf}}_K & & K\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty} \rangle & \\
\vdots \arrow[d, "\varphi"] & & \vdots & \\
\mathbb{P}^{n,\mathrm{ad}}_K \arrow[d, "\varphi"] & (x_0 : \cdots : x_n) \arrow[d, mapsto] & K\langle T_1, \ldots, T_n \rangle \arrow[u, "\varphi^*"'] & T_1^p, \ldots, T_n^p \\
\mathbb{P}^{n,\mathrm{ad}}_K & (x_0^p : \cdots : x_n^p) & K\langle T_1, \ldots, T_n \rangle \arrow[u, "\varphi^*"'] & T_1, \ldots, T_n \arrow[u, mapsto]
\end{tikzcd}$$

Behaves like an inverse limit on topological spaces and étale topoi.

Tilting perfectoid spaces

$$R := K\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle \text{ is a perfectoid } K\text{-algebra.}$$

$$R^\flat := \varprojlim_{x \mapsto x^p} R \cong K^\flat\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle \text{ is the tilt.}$$

$$\sharp : R^\flat \to R \text{ projects onto the initial term}$$

Perfectoid spaces are built from perfectoid rings.

Theorem (Scholze (2012))

Tilting induces an equivalence of categories between perfectoid spaces over K and perfectoid spaces over $K^\flat$. Perfectoid spaces X and $X^\flat$ have isomorphic topological spaces and étale topoi.

Tilting $\mathbb{P}^{n,\mathrm{perf}}_K$

$$\begin{tikzcd}
\mathbb{P}^{n,\mathrm{perf}}_K \arrow[r, dashed] & (\mathbb{P}^{n,\mathrm{perf}}_K)^\flat & \mathbb{P}^{n,\mathrm{perf}}_{K^\flat} \arrow[l, "\cong"'] \\
\vdots \arrow[d, "\varphi"] & & \vdots \arrow[d, "\varphi"] \\
\mathbb{P}^{n,\mathrm{ad}}_K \arrow[d, "\varphi"] & & \mathbb{P}^{n,\mathrm{ad}}_{K^\flat} \arrow[d, "\varphi"] \\
\mathbb{P}^{n,\mathrm{ad}}_K & & \mathbb{P}^{n,\mathrm{ad}}_{K^\flat}
\end{tikzcd}$$

Tilting $\mathbb{P}_K^{n,\mathrm{perf}}$

$$\begin{tikzcd}
\mathbb{P}_K^{n,\mathrm{perf}} \arrow[r, dashed, shift left] & \left(\mathbb{P}_K^{n,\mathrm{perf}}\right)^\flat \arrow[l, dashed, shift left] & \mathbb{P}_{K^\flat}^{n,\mathrm{perf}} \arrow[l, "\cong"'] \\
\vdots \arrow[d, "\varphi"] & & \vdots \arrow[d, shift left, "\varphi"] \\
\mathbb{P}_K^{n,\mathrm{ad}} \arrow[d, "\varphi"] & & \mathbb{P}_{K^\flat}^{n,\mathrm{ad}} \arrow[u, dashed, shift left] \arrow[d, shift left, "\varphi"] \\
\mathbb{P}_K^{n,\mathrm{ad}} & & \mathbb{P}_{K^\flat}^{n,\mathrm{ad}} \arrow[u, dashed, shift left] \arrow[ll, dashed, "\pi"']
\end{tikzcd}$$

$$(x_0^\sharp : \cdots : x_n^\sharp) \longleftarrow\!\shortmid (x_0 : \cdots : x_n)$$

Can we pull back varieties from $\mathbb{P}_K^{n,\mathrm{ad}}$ to $\mathbb{P}_{K^\flat}^{n,\mathrm{ad}}$ and apply Deligne's result?

Scholze's approach to weight-monodromy

Varieties don't pull back to varieties!

Approximation lemma: given a hypersurface $Y \subset \mathbb{P}_K^{n,\mathrm{ad}}$, and a small open neighborhood $U \supset Y$, there is a hypersurface $Y' \subset \pi^{-1}(U)$.

$$\begin{tikzcd}
\mathbb{P}_K^{n,\mathrm{ad}} & \mathbb{P}_{K^\flat}^{n,\mathrm{ad}} \arrow[l, dashed, "\pi"'] \\
U \arrow[u, hook] & \pi^{-1}(U) \arrow[l, dashed, "\pi"'] \arrow[u, hook] \\
Y \arrow[u, hook] & Y' \arrow[u, hook]
\end{tikzcd}$$

Huber: For small enough U, the map on étale cohomology induced by $Y \hookrightarrow U$ is an isomorphism. Get $H^i(Y, \overline{\mathbb{Q}}_\ell) \cong H^i(U, \overline{\mathbb{Q}}_\ell) \hookrightarrow H^i(Y', \overline{\mathbb{Q}}_\ell)$.

Generalizing to abelian varieties

Abelian variety A over $\mathbb{Q}_p$

- Construct perfectoid cover A_∞ of A_K.
- Construct abelian variety $A'/K^\flat$ with $A'_\infty \cong A^\flat_\infty$.
- Prove approximation lemma.

Perfectoid covers of abelian varieties

Theorem (Blakestad, Gvirtz, Heuer, Shchedrina, Shimizu, W., Yao)

Let A be an abelian variety over a perfectoid field K of residue characteristic p with value group contained in $\mathbb{Q}$. There is a perfectoid group A_∞ such that $A_\infty \sim \varprojlim_{[p]} A$.

Already known in the good reduction case

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Let A be an abelian variety over a perfectoid field K of residue characteristic p with value group contained in $\mathbb{Q}$. There is a perfectoid group A_∞ such that $A_\infty \sim \varprojlim_{[p]} A$.

The main idea is to construct an inverse system of formal schemes

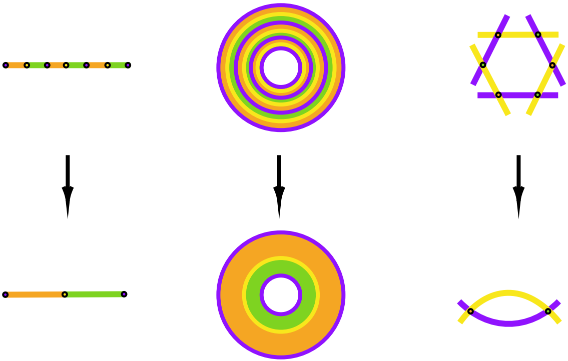
$$\cdots \xrightarrow{[\mathfrak{p}_2]} \mathfrak{A}_1 \xrightarrow{[\mathfrak{p}_1]} \mathfrak{A}_0$$

all flat over K° such that:

- The transition maps are all affine,
- The adic generic fiber of each $\mathfrak{A}_i$ is A, and the adic generic fiber of each morphism is $[p]$, and
- The mod ϖ special fiber of each $[\mathfrak{p}_i]$ factors through relative Frobenius.

$$A_\infty := (\varprojlim_{[\mathfrak{p}_i]} \mathfrak{A}_i)_\eta$$

The tower over the Tate curve



Middle: a system of formal analytic coverings of a Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$.
Right: the tower of special fibers.
Left: the tropicalization, coming from the log map
$\mathbb{G}_m(K) \to \mathbb{R} : x \mapsto \log(|x|_K)$.

Formal models of abelian varieties: Raynaud uniformization

Let A be an abelian variety over a non-archimedean field K, considered as an adic space. Raynaud: after a finite extension of K, can uniformize A

$$\begin{tikzcd} & M \arrow[d, hook] & \\ T \arrow[r] & E \arrow[r] \arrow[d] & B \\ & A & \end{tikzcd}$$

- $T = \mathbb{G}_m^r$ a split torus
- B an abelian variety with good reduction
- E a semi-abelian variety: a product of translation-invariant $\mathbb{G}_m$-torsors on B
- M a lattice in $E(K)$: a discrete subgroup isomorphic to $\mathbb{Z}^r$
- $A \cong E/M$

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Construct a formal model of T, take the pushout to get a formal model of E.

Make sure this model retains M-action, and take the quotient to get a model of A.

Tilting perfectoid covers of abelian varieties

Theorem (Heuer, W.)

After a pro-p extension of K, there is an abelian variety A' over $K^\flat$ such that $A_\infty^\flat \sim \varprojlim_{[p]} A'$.

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Starting with an inverse system of formal models of A/K

$$\cdots \xrightarrow{[\mathfrak{p}_2]} \mathfrak{A}_1 \xrightarrow{[\mathfrak{p}_1]} \mathfrak{A}_0$$

Need to construct $A'/K^\flat$ along with an inverse system of formal models

$$\cdots \xrightarrow{[\mathfrak{p}_2']} \mathfrak{A}_1' \xrightarrow{[\mathfrak{p}_1']} \mathfrak{A}_0'$$

Such that the mod ϖ and mod $\varpi^\flat$ special fibers agree.

Tilting examples

Given B/K with good reduction, we can extend to an abelian scheme over K° with special fiber an abelian scheme $\widetilde{B}$ over $K^\circ/\varpi \cong K^{\flat\circ}/\varpi^\flat$.

Deform $\widetilde{B}$ to an abelian scheme over $K^{\flat\circ}$, take the generic fiber to get $B'/K^\flat$. This is not unique!

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Given a Tate curve $\mathbb{G}_{m,K}/q^{\mathbb{Z}}$, there's a natural candidate for $q' \in \mathbb{G}_{m,K^\flat}$.

$$\sharp : (K^\flat)^\times \to K^\times$$
$$q^\flat \mapsto q$$
$$(K^\flat)^\times/(q^\flat)^{\mathbb{Z}} \leftarrow\!\shortmid K^\times/q^{\mathbb{Z}}$$

A pro-p extension of K is needed to ensure q is in the image of $\sharp$. Again non-unique.

Tilting in general

- Use deformation theory to construct a Raynaud extension $T' \to E' \to B'$ over $K^\flat$ with the correct special fiber.
- Choose a compatible system of pth power roots of M (after a pro-p extension of K), giving a lattice $M \cong M_\infty \subset E_\infty \sim \varprojlim E$.
- Tilt to get a lattice $M_\infty^\flat \subset E_\infty^\flat$ which projects to a lattice $M' \subset E'$.

This gives a candidate E'/M' for A' which will have the correct perfectoid cover. Need to check that this is an abelian variety: that it has an ample line bundle.

Tilting line bundles

Let $L^\times$ be a $\mathbb{G}_m$-torsor on A. This pulls back to a $\mathbb{G}_m$-torsor $L_1^\times$ on A_∞. If we can extend this to a $\mathbb{G}_{m,\infty}$-torsor over A_∞, this will be perfectoid. This can be done if we can extract pth power roots of $L_1^\times$ in $\mathrm{Pic}(A_\infty)$.

$$\mathbb{G}_{m,\infty} \longrightarrow L_\infty^\times \longrightarrow A_\infty$$

$$\begin{tikzcd}
\vdots \arrow[d, "{[p]}"] & \vdots \arrow[d] & \vdots \arrow[d, equal] \\
\mathbb{G}_m \arrow[r] \arrow[d, "{[p]}"] & L_2^\times \arrow[r] \arrow[d] & A_\infty \arrow[d, equal] \\
\mathbb{G}_m \arrow[r] \arrow[d, equal] & L_1^\times \arrow[r] \arrow[d] & A_\infty \arrow[d] \\
\mathbb{G}_m \arrow[r] & L^\times \arrow[r] & A
\end{tikzcd}$$

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Tilting this sequence of perfectoid groups gives a $\mathbb{G}_{m,\infty,K^\flat}$-torsor over $A_\infty^\flat \cong A'_\infty$.

If we've made good choices, this will descend to a line bundle L' on A'.

Hypersurface approximation

- Given a hypersurface $Y \subset A$, there is a line bundle L on A and an element $f \in H^0(A, L)$ with zero locus Y.
- We can put $H^0(A, L) \hookrightarrow \Gamma(L_\infty^\times)$, which is a perfectoid ring tilting to $\Gamma(L_\infty'^\times)$.
- We can approximate f by $g^\sharp$ for some $g \in \Gamma(L_\infty'^\times)$.
- Then approximate g by an element coming from finite level, which defines a hypersurface $Z \subset A'$.

Generalizing to abelian varieties

- Construct perfectoid cover A_∞ of A_K.
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- Prove approximation lemma.

$$
\begin{tikzcd}
 & & (A^\flat_\infty)^\sim_{\text{ét}} \arrow[dll, "q_1'"'] \arrow[drr, "q_1"] & & \\
(A'_{\mathbb{C}_p^\flat})^\sim_{\text{ét}} & & (q_1^{-1}(U)_{\mathbb{C}_p^\flat})^\sim_{\text{ét}} \arrow[u] \arrow[dll] \arrow[drr] & & (A_{\mathbb{C}_p})^\sim_{\text{ét}} \\
(U'_{\mathbb{C}_p^\flat})^\sim_{\text{ét}} \arrow[u] & & & & (U_{\mathbb{C}_p})^\sim_{\text{ét}} \arrow[u] \\
(Z_{\mathbb{C}_p^\flat})^\sim_{\text{ét}} \arrow[u] & & & & (Y_{\mathbb{C}_p})^\sim_{\text{ét}}. \arrow[u, "\cong"]
\end{tikzcd}
$$

Conclusion

Theorem (W.)

The weight-monodromy conjecture holds for complete intersections in abelian varieties.

$$\begin{tikzcd}
A_\infty & A'_\infty \arrow[l, dashed] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
A \arrow[d, "{[p]}"] & A' \arrow[d, "{[p]}"] \\
A & A'
\end{tikzcd}$$

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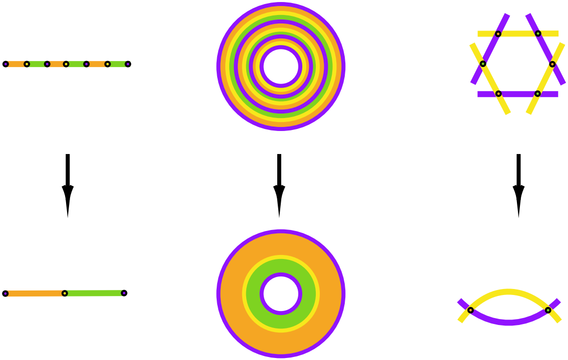
A \arrow[dd, "{[p]}"] & A' \arrow[dd, "{[p]}"] \arrow[dr, dashed, leftrightarrow, "\cong"] & \\
 & & A'/A'[p]^0 \arrow[dl, "\text{fét}"] \\
A & A' &
\end{tikzcd}$$

Use the trace map on the finite étale part.

Thank you!



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