Smoothness of cohomology sheaves of stacks of shtukas

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Introduction

In this talk, we will

- recall the definition of the stacks of shtukas and the cohomology sheaves
- prove the smoothness property of the cohomology sheaves (to appear soon)

Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$, $\operatorname{char} \mathbb{F}_q = p$. Let F be its function field.
Let G be a reductive group over F.

In the talk : to simplify, we only consider the case without level structure and we suppose that G is split.

Let $\widehat{G}$ be the Langlands dual group of G over $\mathbb{Q}_\ell$, where $\ell \neq p$.

Stacks of shtukas : an example

$I = \{1, 2\}$, $W = St \boxtimes St^*$, St standard representation of GL_n, St^* the dual of St. For any S affine scheme over $\mathbb{F}_q$, we denote by $\mathrm{Frob}_S : S \to S$ the absolute Frobenius morphism over $\mathbb{F}_q$.

Drinfeld's stack of right shtukas :

$$\mathrm{Cht}^{(1,2)}_{G,I,W}(S) := \{x_1, x_2 \in X(S), \mathcal{G}_0, \mathcal{G}_1 : \text{rk } n \text{ vector bundles on } X \times_{\mathbb{F}_q} S,$$

$$\mathcal{G}_0 \overset{\phi_1}{\hookrightarrow} \mathcal{G}_1 \overset{\phi_2}{\hookleftarrow} (\mathrm{Id}_X \times \mathrm{Frob}_S)^* \mathcal{G}_0 \ \text{ s.t.}$$

$$\mathcal{G}_1 / \mathcal{G}_0 \text{ is an invertible sheaf on the graph of } x_1,$$

$$\mathcal{G}_1 / (\mathrm{Id}_X \times \mathrm{Frob}_S)^* \mathcal{G}_0 \text{ is an invertible sheaf on the graph of } x_2\}.$$

Drinfeld's stack of left shtukas :

$$\mathrm{Cht}^{(2,1)}_{G,I,W}(S) := \{x_1, x_2 \in X(S), \mathcal{G}'_0, \mathcal{G}'_1 : \text{rk } n \text{ vector bundles on } X \times_{\mathbb{F}_q} S,$$

$$\mathcal{G}'_0 \overset{\phi'_1}{\hookleftarrow} \mathcal{G}'_1 \overset{\phi'_2}{\hookrightarrow} (\mathrm{Id}_X \times \mathrm{Frob}_S)^* \mathcal{G}'_0 \ \text{ s.t.}$$

$$\mathcal{G}'_0 / \mathcal{G}'_1 \text{ is an invertible sheaf on the graph of } x_2,$$

$$(\mathrm{Id}_X \times \mathrm{Frob}_S)^* \mathcal{G}'_0 / \mathcal{G}'_1 \text{ is an invertible sheaf on the graph of } x_1\}.$$

Stacks of shtukas : in general

Let $I = \{1, 2, \cdots, k\}$ be a finite set. Let W be a finite dim $\mathbb{Q}_\ell$-linear representation of $\widehat{G}^I$. Suppose $W = \boxtimes_{i \in I} W_i$, with W_i irreducible representation of $\widehat{G}$ of highest weight λ_i.

Varshavsky defined the stack of shtukas associated to I , W and order $(1, 2, \cdots, k)$:

$$\mathrm{Cht}_{G,I,W}^{(1,2,\cdots,k)}(S) := \{(x_i)_{i \in I} \in X^I(S),$$

$$\mathcal{G}_0, \mathcal{G}_1, \cdots, \mathcal{G}_{k-1} : G\text{-bundles on } X \times_{\mathbb{F}_q} S,$$

$$\mathcal{G}_0 \xrightarrow{\phi_1} \mathcal{G}_1 \xrightarrow{\phi_2} \cdots \dashrightarrow \mathcal{G}_{k-1} \xrightarrow{\phi_k} (\mathrm{Id}_X \times \mathrm{Frob}_S)^* \mathcal{G}_0 \quad \text{where}$$

$$\phi_i \text{ isom outside } x_i, \text{ relative position of } \mathcal{G}_{i-1} \text{ and } \mathcal{G}_i \text{ at } x_i \text{ bounded by } \lambda_i\}$$

A shtuka is a S-point of the stack of shtukas. The x_i are called the paws of the shtuka.

We can also define stack of shtukas associated to order $(2, 3, \cdots, k, 1)$, $(3, 4, \cdots, k, 1, 2)$ etc... In the following, we will omit the index $(1, 2, \cdots, k)$ when there is no confusion. In any way, the cohomology sheaves is independent of the order.

$\mathrm{Cht}_{G,I,W}$ is a Deligne-Mumford stack locally of finite type.

For those familiar with Bun_G, an equivalent definition : the stack of shtukas is the following fiber product

$$\begin{tikzcd} \mathrm{Cht}_{G,I,W} \arrow[r] \arrow[d] & \mathrm{Bun}_G \arrow[d, "(\mathrm{Id},\mathrm{Frob})"] \\ \mathrm{Hecke}_{G,I,W} \arrow[r] & \mathrm{Bun}_G \times \mathrm{Bun}_G \end{tikzcd}$$

$$((x_i), \mathcal{G}_0 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{k-1}} \mathcal{G}_{k-1} \xrightarrow{\phi_k} \mathcal{G}_k) \qquad \mapsto \qquad (\mathcal{G}_0, \mathcal{G}_k)$$

Satake sheaf over stack of shtukas

We have the morphism of paws

$$\mathfrak{p} : \mathrm{Cht}_{G,I,W} \to X^I$$

In general, the stack of shtukas $\mathrm{Cht}_{G,I,W}$ is not smooth. We have a canonical perverse sheaf $\mathrm{Sat}_{G,I,W}$ over $\mathrm{Cht}_{G,I,W}$, which comes from the geometric Satake equivalence (Mirkovic-Vilonen).

When W is irreducible, $\mathrm{Sat}_{G,I,W}$ is isomorphic to the $\mathbb{Q}_\ell$-coefficient IC-sheaf of $\mathrm{Cht}_{G,I,W}$ (relative to X^I). Example : when $\mathrm{Cht}_{G,I,W}$ is smooth, $\mathrm{Sat}_{G,I,W} = \mathbb{Q}_\ell[d]$, where $d = \dim \mathrm{Cht}_{G,I,W} - \dim X^I$.

$$\mathrm{Cht}_{G,I,W\oplus W'} := \mathrm{Cht}_{G,I,W} \bigcup \mathrm{Cht}_{G,I,W'}$$

$$\mathrm{Sat}_{G,I,W\oplus W'} := \mathrm{Sat}_{G,I,W} \bigoplus \mathrm{Sat}_{G,I,W'}$$

Harder-Narasimhan stratification

To simply the notation, suppose that G is semisimple. The stack of shtukas $\mathrm{Cht}_{G,I,W}$ is locally of finite type but not necessarily of finite type.

We have the Harder-Narasimhan stratification : for any μ dominant coweight of G, we have

$$\begin{tikzcd} \mathrm{Cht}_{G,I,W}^{\leq\mu} \arrow[r, hook, "\text{open}"] \arrow[d] & \mathrm{Cht}_{G,I,W} \arrow[d] \\ \mathrm{Bun}_G^{\leq\mu} \arrow[r, hook, "\text{open}"] & \mathrm{Bun}_G \end{tikzcd}$$

where $\mathrm{Bun}_G^{\leq\mu} = \{\mathcal{G}_0, \text{ the slope of } \mathcal{G}_0 \leq \mu\}$.
The open substack $\mathrm{Cht}_{G,I,W}^{\leq\mu}$ is of finite type. And we have

$$\mathrm{Cht}_{G,I,W} = \bigcup_{\mu\in\Lambda} \mathrm{Cht}_{G,I,W}^{\leq\mu}$$

Cohomology sheaves of the stack of shtukas

Recall that we have the morphism of paws $\mathfrak{p} : \mathrm{Cht}_{G,I,W} \to X^I$. We define the truncated cohomology sheaf on degree $j \in \mathbb{Z}$

$$\mathcal{H}^{j,\leq\mu}_{G,I,W} := R^j\mathfrak{p}_!(\mathrm{Sat}_{G,I,W}|_{\mathrm{Cht}^{\leq\mu}_{G,I,W}})$$

It is a constructible $\mathbb{Q}_\ell$-sheaf over X^I. Cohomology sheaves are concentrated in degree $j \in [-d, d]$ where $d = \dim \mathrm{Cht}_{G,I,W} - \dim X^I$.

For $\mu_1 \leq \mu_2$, we have an open immension

$$\mathrm{Cht}^{\leq\mu_1}_{G,I,W} \hookrightarrow \mathrm{Cht}^{\leq\mu_2}_{G,I,W}$$

It induces a morphism of sheaves

$$\mathcal{H}^{j,\leq\mu_1}_{G,I,W} \to \mathcal{H}^{j,\leq\mu_2}_{G,I,W}.$$

We define the degree j cohomology sheaf

$$\mathcal{H}^{j}_{G,I,W} := \varinjlim_{\mu} \mathcal{H}^{j,\leq\mu}_{G,I,W}.$$

Let η_I be the generic point of X^I. Let $\overline{\eta_I}$ be a geometric point over η_I. We define the cohomology group

$$H^j_{G,I,W} := \mathcal{H}^j_{G,I,W}\Big|_{\overline{\eta_I}}$$

When $I = \emptyset$ (empty set), $W = \mathbf{1}$ (trivial representation), we have $\mathrm{Cht}_{G,\emptyset,\mathbf{1}} = \mathrm{Bun}_G(\mathbb{F}_q)$ and $H^0_{G,\emptyset,\mathbf{1}} = C_c(\mathrm{Bun}_G(\mathbb{F}_q), \mathbb{Q}_\ell)$.

In general, $H^j_{G,I,W}$ is a $\mathbb{Q}_\ell$-vector space of possibly infinite dimension, equiped with

- an action of the Hecke algebra $\mathscr{H}_G := C_c(G(\mathbb{O})\backslash G(\mathbb{A})/G(\mathbb{O}), \mathbb{Q}_\ell)$ by the Hecke correspondences
- an action of $\mathrm{Weil}(\eta_I, \overline{\eta_I})$ (evident)
- an action of the partial Frobenius morphisms (one of the key properties of stack of shtukas, will be defined in the next page)

Partial Frobenius morphisms : an example

Consider Drinfeld's stacks of shtukas. Let $I = \{1, 2\}$, $W = St \boxtimes St^*$.
Denote by ${}^\tau\mathcal{G} := (\mathrm{Id}_X \times \mathrm{Frob}_S)^*\mathcal{G}$ and $\mathrm{Frob} : X \to X$ the absolute Frobenius.

$$(\mathcal{G}_0 \overset{\phi_1}{\hookrightarrow} \mathcal{G}_1 \overset{\phi_2}{\hookleftarrow} {}^\tau\mathcal{G}_0) \mapsto (\mathcal{G}_1 \overset{\phi_2}{\hookleftarrow} {}^\tau\mathcal{G}_0 \overset{{}^\tau\phi_1}{\hookrightarrow} {}^\tau\mathcal{G}_1) \mapsto ({}^\tau\mathcal{G}_0 \overset{{}^\tau\phi_1}{\hookrightarrow} {}^\tau\mathcal{G}_1 \overset{{}^\tau\phi_2}{\hookleftarrow} {}^\tau{}^\tau\mathcal{G}_0)$$

$$\begin{tikzcd}
\mathrm{Cht}_{G,I,W}^{(1,2)} \arrow[r, "\mathrm{Frob}_{\{1\}}"] \arrow[d, "\mathfrak{p}"] & \mathrm{Cht}_{G,I,W}^{(2,1)} \arrow[r, "\mathrm{Frob}_{\{2\}}"] \arrow[d, "\mathfrak{p}"] & \mathrm{Cht}_{G,I,W}^{(1,2)} \arrow[d, "\mathfrak{p}"] \\
X^2 \arrow[r, "\mathrm{Frob}_{\{1\}}"] & X^2 \arrow[r, "\mathrm{Frob}_{\{2\}}"] & X^2
\end{tikzcd}$$

$$(x_1, x_2) \mapsto (\mathrm{Frob}(x_1), x_2) \mapsto (\mathrm{Frob}(x_1), \mathrm{Frob}(x_2))$$

$$\mathrm{Frob}_{\{2\}} \circ \mathrm{Frob}_{\{1\}} = \text{ Frobenius total sur } \mathrm{Cht}_{G,I,W}^{(1,2)}$$

Partial Frobenius morphisms : in general

In general, let $I = \{1, 2, \cdots, k\}$.

$$\begin{tikzcd} \mathrm{Cht}_{G,I,W}^{(1,2,\cdots,k)} \arrow[r, "\mathrm{Frob}_{\{1\}}"] \arrow[d, "\mathfrak{p}"] & \mathrm{Cht}_{G,I,W}^{(2,\cdots,k,1)} \arrow[d, "\mathfrak{p}"] \\ X^I \arrow[r, "\mathrm{Frob}_{\{1\}}"] & X^I \end{tikzcd}$$

We have a canonical morphism :

$$\mathrm{Frob}_{\{1\}}^* \mathrm{Sat}_{G,I,W}^{(2,\cdots,k,1)} \xrightarrow{\sim} \mathrm{Sat}_{G,I,W}^{(1,2,\cdots,k)}$$

Cohomological correspondence induces a partial Frobenius morphism :

$$F_{\{1\}} : \mathrm{Frob}_{\{1\}}^* \mathcal{H}_{G,I,W}^{j,\leq\mu} \to \mathcal{H}_{G,I,W}^{j,\leq\mu+\kappa}$$

No index $(1, 2, \cdots, k)$, because of the fact : the cohomology sheaves are independent of the order, because the following morphism π is small.

$$\mathrm{Cht}_{G,I,W}^{(1,2,\cdots,k)} \xrightarrow{\pi} \mathrm{Cht}_{G,I,W}^{I} \to X^I$$

$$((x_i)_{i\in I}, \mathcal{G}_0 \dashrightarrow \mathcal{G}_1 \dashrightarrow \cdots \dashrightarrow {}^\tau\mathcal{G}_0) \mapsto ((x_i)_{i\in I}, \mathcal{G}_0 \dashrightarrow {}^\tau\mathcal{G}_0) \mapsto (x_i)_{i\in I}$$

Similarly, we have $F_{\{2\}}, \cdots, F_{\{k\}}$. The composition $F_{\{1\}} \circ \cdots \circ F_{\{k\}}$ is the total Frobenius morphism (composed with an augmentation of μ).

On the inductive limite, we have

$$F_{\{i\}} : \mathrm{Frob}_{\{i\}}^* \mathcal{H}_{G,I,W}^{j} \xrightarrow{\sim} \mathcal{H}_{G,I,W}^{j}$$

Remark : the action of $\mathrm{Weil}(\eta_I, \overline{\eta_I})$ preserves $\mathcal{H}_{G,I,W}^{j,\leq\mu}\Big|_{\overline{\eta_I}}$, however, the action of Hecke algebra and the action of the partial Frobenius morphisms do NOT preserve $\mathcal{H}_{G,I,W}^{j,\leq\mu}\Big|_{\overline{\eta_I}}$.

Work of V. Lafforgue

$$X^I \leftarrow \eta_I \leftarrow \overline{\eta_I} \qquad\qquad \begin{array}{ccc} \pi_1(\eta_I, \overline{\eta_I}) & \longrightarrow & \widehat{\mathbb{Z}} \\ \downarrow & & \downarrow \\ \pi_1(\eta, \overline{\eta})^I & \longrightarrow & \widehat{\mathbb{Z}}^I \end{array}$$

$$X \leftarrow \eta \leftarrow \overline{\eta}$$

A lemma of Drinfeld : finite type $\mathbb{Z}_\ell$-module equiped with an action of $\pi_1(\eta_I, \overline{\eta_I})$ and an action of the partial Frobenius morphisms $\Rightarrow$ action of $\pi_1(\eta, \overline{\eta})^I = \mathrm{Gal}(\overline{F}/F)^I$ (where F is the function field of X)

V. Lafforgue defined Hecke-finite cohomology $H^{j,\mathrm{Hf}}_{G,I,W} \subset H^j_{G,I,W}$. By the Eichler-Shimura relation, $H^{j,\mathrm{Hf}}_{G,I,W}$ is an inductive limite of finite type $\mathbb{Z}_\ell$-modules. By Drinfeld's lemma, it is equipped with an action of $\mathrm{Gal}(\overline{F}/F)^I$.
Using this, and the creation and annihilation operators, V. Lafforgue constructed the excursion operators on the space of cuspidal automorphic forms and proved the "automorphic to Galois" direction of the Langlands correspondence.

Smoothness

In the following part of the talk, we will
(1) use a variant of Drinfeld's lemma to prove

Proposition

(a) $\mathcal{H}^j_{G,I,W}\Big|_{\overline{\eta_I}}$ is equiped with an action of $\mathrm{Weil}(\eta, \overline{\eta})^I = \mathrm{Weil}(\overline{F}/F)^I$.
(b) the restriction $\mathcal{H}^j_{G,I,W}\Big|_{(\overline{\eta})^I}$ is constant over $(\overline{\eta})^I := \overline{\eta} \times_{\overline{\mathbb{F}_q}} \cdots \times_{\overline{\mathbb{F}_q}} \overline{\eta}$.

(2) use this proposition, the creation and annihilation operators to prove

Theorem

The $\mathbb{Q}_\ell$-sheaf $\mathcal{H}^j_{G,I,W}$ is ind-smooth over X^I.

Corollary

The action of $\mathrm{Weil}(\overline{F}/F)^I$ on $\mathcal{H}^j_{G,I,W}\Big|_{\overline{\eta_I}}$ factors through $\mathrm{Weil}(X, \eta)^I$

Illustration for Proposition

(This is not our case, just to see what happens in a simple case.)

Let $\mathcal{F}$ be a constructible $\mathbb{Q}_\ell$-sheaf over X^I equiped with an action of the partial Frobenius morphisms.

Lemma 1 (Drinfeld)

The finite dim $\mathbb{Q}_\ell$-vector space $\mathcal{F}|_{\overline{\eta_I}}$ is equiped with an action of $\mathrm{Weil}(\eta, \overline{\eta})^I$.

$\Rightarrow$ the action of $\mathrm{Weil}((\overline{\eta})^I, \overline{\eta_I})$ on $\mathcal{F}|_{\overline{\eta_I}}$ is trivial.

Lemma 2 (Drinfeld, Eike Lau)

$\mathcal{F}$ is smooth over U^I for some dense open subscheme U of X.

Lemma 1 + Lemma 2 $\Rightarrow \mathcal{F}|_{(\overline{\eta})^I}$ is constant over $(\overline{\eta})^I$.

Proof of Proposition

Recall that $\mathcal{H}^{j}_{G,I,W} := \varinjlim_{\mu} \mathcal{H}^{j,\leq\mu}_{G,I,W}$. The inductive limite $\mathcal{H}^{j}_{G,I,W}$ has an action of the partial Frobenius morphisms but may not be constructible. Each $\mathcal{H}^{j,\leq\mu}_{G,I,W}$ is constructible but does not have an action of the partial Frobenius morphisms.

Solution : we have $\mathcal{H}^{j}_{G,I,W}\Big|_{\overline{\eta_I}} := \varinjlim_{\mu} \mathfrak{M}_{\mu}$ with

$$\mathfrak{M}_{\mu} := \sum_{(n_i)_{i\in I}\in\mathbb{N}^I} (\otimes_{i\in I}\mathscr{H}_{G,v_i}) \cdot \big(\prod_{i\in I} \mathrm{Frob}^{n_i}_{\{i\}} \mathcal{H}^{j,\leq\mu}_{G,I,W}\big)\Big|_{\overline{\eta_I}}$$

where v_i are closed points of X (chosen such that $\times_{i\in I} v_i$ is included in the smooth locus of $\mathcal{H}^{j,\leq\mu}_{G,I,W}$) and $\mathscr{H}_{G,v_i}$ is the local Hecke algebra on v_i.

By the Eichler-Shimura relations, the sum is in fact over a finite number of $(n_i)_{i\in I}$. Thus each $\mathfrak{M}_{\mu}$ is a module of finite type over a Hecke algebra.

Lemma 1' (Drinfeld)

The Hecke algebra module of finite type $\mathfrak{M}_\mu|_{\overline{\eta_I}}$ is equiped with an action of $\mathrm{Weil}(\eta, \overline{\eta})^I$.

$\Rightarrow$ the action of $\mathrm{Weil}((\overline{\eta})^I, \overline{\eta_I})$ on $\mathfrak{M}_\mu|_{\overline{\eta_I}}$ is trivial.

However, we do not have a generalisation of Lemma 2. We prove by other reasons that $\mathcal{H}^j_{G,I,W}$ is smooth over $(\overline{\eta})^I$. The proof is similar to V. Lafforgue's proof that $\mathcal{H}^j_{G,I,W}\Big|_{\Delta(\overline{\eta})} \to \mathcal{H}^j_{G,I,W}\Big|_{\overline{\eta_I}}$ is an isomorphism.

$\Rightarrow \mathcal{H}^j_{G,I,W}\Big|_{(\overline{\eta})^I}$ is constant over $(\overline{\eta})^I$.

$\square$

Proof of smoothness : example of I singleton

Let $I = \{1\}$ be a singleton. Let W be a representation of $\widehat{G}$. We have a cohomology sheaf $\mathcal{H}^j_{G,\{1\},W}$ over X.

For any geometric point $\overline{v}$ of X (over a closed point v) and any specialization map $\mathfrak{sp} : \overline{\eta} \to \overline{v}$, we have an induced morphism

$$\mathfrak{sp}^* : \mathcal{H}^j_{G,\{1\},W}\Big|_{\overline{v}} \to \mathcal{H}^j_{G,\{1\},W}\Big|_{\overline{\eta}}$$

We want to prove that $\mathfrak{sp}^*$ is an isomorphism. This is what we mean "ind-smooth" over X.

Idea : construct an inverse of $\mathfrak{sp}^*$ using some creation and annihilation operators.

Reminder about creation operator

Let W^* be the dual representation of W. Denote by $\mathbf{1}$ the trivial representation of $\widehat{G}$. Let $\delta : \mathbf{1} \to W^* \otimes W, 1 \mapsto \sum_k e_k^* \otimes e_k$. Denote by $\mathbb{Q}_{\ell X}$ the constant sheaf over X.

The creation operator $\mathcal{C}_\delta^{\sharp,\{2,3\}}$ is defined to be the composition of morphisms of sheaves over $X \times X$

$$\begin{tikzcd} \mathcal{H}^j_{\{1\},W} \boxtimes \mathbb{Q}_{\ell X} \arrow[r, "\simeq"] & \mathcal{H}^j_{\{1,0\},W\boxtimes\mathbf{1}} \arrow[r, "\mathcal{H}(\mathrm{Id}_W \boxtimes \delta)", "\text{functoriality}"'] & \mathcal{H}^j_{\{1,0\},W\boxtimes(W^*\otimes W)} \arrow[d, "\simeq"'] \\ & & \mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{X\times\Delta^{\{2,3\}}(X)} \end{tikzcd}$$

where $X \times \Delta^{\{2,3\}}(X)$ is the image of

$$X \times X \xrightarrow{(\mathrm{Id},\Delta^{\{2,3\}})} X \times X \times X$$

An example : $G = GL_2$, Drinfeld's stacks of shtukas, $x \in X$

$$\begin{tikzcd} & \mathcal{G}_1 \arrow[dl, hook, "\phi_1"'] \arrow[dr, hook, "\phi_2=\phi_1"] & \\ \mathcal{G}_0 \arrow[rr, "\sim"] & & {}^\tau\mathcal{G}_0 \end{tikzcd} \qquad \mapsto \qquad (\mathcal{G}_0 \hookleftarrow \mathcal{G}_1 \hookrightarrow {}^\tau\mathcal{G}_0)$$

$$\begin{tikzcd} & \mathbb{P}^1 \arrow[rr] \arrow[d] & & \mathrm{Cht}^{(2,3)}_{\{2,3\},St^*\boxtimes St}\Big|_{\Delta(x)} \arrow[d, "\pi"] \\ \mathrm{Bun}_G(\mathbb{F}_q) \times x = & \mathrm{Cht}_{\{0\},\mathbf{1}}\Big|_x \arrow[r, hook] & \mathrm{Cht}_{\{0\},St^*\otimes St}\Big|_x \arrow[r, "\simeq"] & \mathrm{Cht}_{\{2,3\},St^*\boxtimes St}\Big|_{\Delta(x)} \end{tikzcd}$$

$$(\mathcal{G}_0 \xrightarrow{\sim} {}^\tau\mathcal{G}_0) \qquad \mapsto \qquad (\mathcal{G}_0 \dashrightarrow {}^\tau\mathcal{G}_0, \text{ s.t. } \exists\, \mathcal{G}_1 \ldots)$$

Reminder about annihilation operator

Let $\mathrm{ev} : W \otimes W^* \to \mathbf{1}$ be the evaluation map.

The annihilation operator $\mathcal{C}_{\mathrm{ev}}^{\flat,\{1,2\}}$ is defined to be the composition of morphisms of sheaves over $X \times X$

$$\begin{array}{l} \mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2\}}(X)\times X} \\ \qquad\qquad \Big\downarrow \simeq \\ \qquad \mathcal{H}^j_{\{0,3\},(W\otimes W^*)\boxtimes W} \xrightarrow[\text{functoriality}]{\mathcal{H}(\mathrm{ev}\boxtimes \mathrm{Id}_W)} \mathcal{H}^j_{\{0,3\},\mathbf{1}\boxtimes W} \xrightarrow{\simeq} \mathbb{Q}_{\ell X} \boxtimes \mathcal{H}^j_{\{3\},W} \end{array}$$

where $\Delta^{\{1,2\}}(X) \times X$ is the image of

$$X \times X \xrightarrow{(\Delta^{\{1,2\}},\mathrm{Id})} X \times X \times X$$

Construction of an inverse of $\mathfrak{sp}^*$

We construct

$$\begin{tikzcd}
\mathcal{H}^{j}_{G,\{1\},W}\Big|_{\overline{\eta}} \otimes \mathbb{Q}_\ell\Big|_{\overline{v}} \arrow[d, "\mathcal{C}^{\sharp,\{2,3\}}_{\delta}"', "\text{creation operator restricted to } \overline{\eta}\times\overline{v}"] \\
\mathcal{H}^{j}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\overline{\eta}\times\Delta^{\{2,3\}}(\overline{v})} \arrow[d, "\mathfrak{sp}^*_{\{2\}}"', "\text{canonical morphism}"] \\
\mathcal{H}^{j}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2\}}(\overline{\eta})\times\overline{v}} \arrow[d, "\mathcal{C}^{\flat,\{1,2\}}_{\mathrm{ev}}"', "\text{annihilation operator restricted to } \overline{\eta}\times\overline{v}"] \\
\mathbb{Q}_\ell\Big|_{\overline{\eta}} \otimes \mathcal{H}^{j}_{G,\{3\},W}\Big|_{\overline{v}}
\end{tikzcd}$$

Now explain the construction of the canonical morphism $\mathfrak{sp}^*_{\{2\}}$.

A general lemma

Let S be a trait. Fix

$$\overline{s} = s \to S \leftarrow \eta \leftarrow \overline{\eta}$$

Lemma (S)

Let $\mathcal{G}$ be a (ind-constructible $\mathbb{Q}_\ell -$) sheaf over S. Then $\mathcal{G}$ is given by

$$(\mathcal{G}|_{\overline{s}},\ \mathcal{G}|_{\overline{\eta}},\ \phi : \mathcal{G}|_{\overline{s}} \to \mathcal{G}|_{\overline{\eta}}\ \mathrm{Gal}(\overline{\eta}/\eta)\text{-equivariant})$$

Lemma($S \times S$)

Let $\mathcal{G}$ be a (ind-constructible $\mathbb{Q}_\ell-$) sheaf over $S \times S$. Suppose that $\mathcal{G}|_{\overline{\eta}\times\overline{\eta}}$ is constant. Then $\mathcal{G}$ is given by a $\mathrm{Gal}(\overline{\eta}/\eta)^2$-equivariant commutative diagram

$$\begin{tikzcd}
\mathcal{G}|_{\overline{s}\times\overline{s}} \arrow[r] \arrow[d] & \mathcal{G}|_{\overline{\eta}\times\overline{s}} \arrow[d] \\
\mathcal{G}|_{\overline{s}\times\overline{\eta}} \arrow[r] & \Gamma(\overline{\eta}\times\overline{\eta}, \mathcal{G})
\end{tikzcd}$$

Lemma$(S \times S \times S)$

Let $\mathcal{G}$ be a (ind-constructible $\mathbb{Q}_\ell-$) sheaf over $S \times S \times S$. Suppose that $\mathcal{G}|_{\overline{\eta}\times\overline{\eta}\times\overline{\eta}}$, $\mathcal{G}|_{\overline{\eta}\times\overline{\eta}\times\overline{s}}$, $\mathcal{G}|_{\overline{\eta}\times\overline{s}\times\overline{\eta}}$ and $\mathcal{G}|_{\overline{s}\times\overline{\eta}\times\overline{\eta}}$ are constant. Then $\mathcal{G}$ is given by a $\mathrm{Gal}(\overline{\eta}/\eta)^3$-equivariant commutative diagram

$$
\begin{tikzcd}
 & \mathcal{G}|_{\overline{s}\times\overline{s}\times\overline{s}} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{G}|_{\overline{\eta}\times\overline{s}\times\overline{s}} \arrow[dl] \arrow[dd, "\mathfrak{sp}^*_{\{2\}}"] \\
\mathcal{G}|_{\overline{s}\times\overline{s}\times\overline{\eta}} \arrow[rr] \arrow[dd] & & \Gamma(\overline{\eta}\times\overline{s}\times\overline{\eta}, \mathcal{G}) \arrow[dd] & \\
 & \mathcal{G}|_{\overline{s}\times\overline{\eta}\times\overline{s}} \arrow[rr] \arrow[dl] & & \Gamma(\overline{\eta}\times\overline{\eta}\times\overline{s}, \mathcal{G}) \arrow[dl] \\
\Gamma(\overline{s}\times\overline{\eta}\times\overline{\eta}, \mathcal{G}) \arrow[rr] & & \Gamma(\overline{\eta}\times\overline{\eta}\times\overline{\eta}, \mathcal{G}) &
\end{tikzcd}
$$

Examples : $\mathcal{G} = \mathcal{G}_1 \boxtimes \mathcal{G}_2 \boxtimes \mathcal{G}_3$, with $\mathcal{G}_i$ a sheaf over S.

Construction of morphism $\mathfrak{sp}^*_{\{2\}}$

Recall that we have

Proposition

For any $I = I_1 \sqcup I_2$ and any $\overline{v}$, the restriction $\mathcal{H}^j_{G,I,W}\Big|_{(\overline{\eta})^{I_1} \times (\overline{v})^{I_2}}$ is constant over $(\overline{\eta})^{I_1} \times (\overline{v})^{I_2}$.

(When I_2 is empty, it is the proposition that we saw before. For general I_2 the argument is samilar.)

Applying the Lemma($S \times S \times S$) to $S =$ strict henselisation of X on $\overline{v}$ and $\mathcal{G} = \mathcal{H}^j_{\{1,2,3\}, W \boxtimes W^* \boxtimes W}$, we construct the canonical morphism $\mathfrak{sp}^*_{\{2\}}$.

In the following we want to show that the morphism

$$\mathcal{H}^j_{G,\{1\},W}\Big|_{\overline{\eta}} \xrightarrow{\mathcal{C}^{\{1,2\},\flat}_{\mathrm{ev}} \circ \mathfrak{sp}^*_{\{2\}} \circ \mathcal{C}^{\sharp,\{2,3\}}_{\delta}} \mathcal{H}^j_{G,\{3\},W}\Big|_{\overline{v}}$$

that we just constructed is the inverse of $\mathfrak{sp}^*$.

Reminder about the "Zorro" lemma

Note that the composition

$$W \otimes \mathbb{Q}_\ell \xrightarrow{Id \otimes \delta} W \otimes W^* \otimes W \xrightarrow{\mathrm{ev} \otimes Id} \mathbb{Q}_\ell \otimes W$$

is the identity.

By the functoriality, we have

"Zorro" lemma

The composition of morphisms of sheaves over X :

$$\mathcal{H}^j_{\{1\},W} \otimes \mathbb{Q}_\ell \xrightarrow{\mathcal{C}^{\sharp,\{2,3\}}_{\delta}} \mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2,3\}}(X)} \xrightarrow{\mathcal{C}^{\flat,\{1,2\}}_{\mathrm{ev}}} \mathbb{Q}_\ell \otimes \mathcal{H}^j_{\{3\},W}$$

is the identity.

Injectivity of $\mathfrak{sp}^*$

The following diagram is commutative

$$
\begin{tikzcd}
\mathcal{H}^j_{\{1\},W}\Big|_{\overline{v}} \otimes \mathbb{Q}_\ell\Big|_{\overline{v}} \arrow[r, "\mathfrak{sp}^*"] \arrow[d, "\mathcal{C}^{\sharp,\{2,3\}}_{\delta}"] & \mathcal{H}^j_{\{1\},W}\Big|_{\overline{\eta}} \otimes \mathbb{Q}_\ell\Big|_{\overline{v}} \arrow[d, "\mathcal{C}^{\sharp,\{2,3\}}_{\delta}"] \\
\mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2,3\}}(\overline{v})} \arrow[r, "\mathfrak{sp}^*_{\{1\}}"] \arrow[dr, "\mathfrak{sp}^*_{\{1,2\}}"'] \arrow[dd, "\mathcal{C}^{\flat,\{1,2\}}_{\mathrm{ev}}"'] & \mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\overline{\eta}\times\Delta^{\{2,3\}}(\overline{v})} \arrow[d, "\mathfrak{sp}^*_{\{2\}}"] \\
 & \mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2\}}(\overline{\eta})\times\overline{v}} \arrow[d, "\mathcal{C}^{\flat,\{1,2\}}_{\mathrm{ev}}"'] \\
\mathbb{Q}_\ell\Big|_{\overline{v}} \otimes \mathcal{H}^j_{\{3\},W}\Big|_{\overline{v}} \arrow[r, "\mathrm{Id}", "\simeq"'] & \mathbb{Q}_\ell\Big|_{\overline{\eta}} \otimes \mathcal{H}^j_{\{3\},W}\Big|_{\overline{v}}
\end{tikzcd}
$$

By "Zorro" lemma, the composition of the left vertical morphisms is the identity. Thus $\mathfrak{sp}^*$ is injective.

Surjectivity of $\mathfrak{sp}^*$

The following diagram is commutative

$$\begin{tikzcd}
\mathcal{H}^{j}_{\{1\},W}\Big|_{\overline{\eta}} \otimes \mathbb{Q}_\ell\Big|_{\overline{v}} \arrow[r, "\simeq"'] \arrow[d, "\mathcal{C}_{\delta}^{\sharp,\{2,3\}}"] & \mathcal{H}^{j}_{\{1\},W}\Big|_{\overline{\eta}} \otimes \mathbb{Q}_\ell\Big|_{\overline{\eta}} \arrow[d, "\mathcal{C}_{\delta}^{\sharp,\{2,3\}}"] \\
\mathcal{H}^{j}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\overline{\eta}\times_{\overline{\mathbb{F}_q}}\Delta^{\{2,3\}}(\overline{v})} \arrow[r, "\mathfrak{sp}^*_{\{2,3\}}"] \arrow[d, "\mathfrak{sp}^*_{\{2\}}"] & \mathcal{H}^{j}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2,3\}}(\overline{\eta})} \arrow[dd, "\mathcal{C}_{\mathrm{ev}}^{\flat,\{1,2\}}"'] \\
\mathcal{H}^{j}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2\}}(\overline{\eta})\times_{\overline{\mathbb{F}_q}}\overline{v}} \arrow[ur, "\mathfrak{sp}^*_{\{3\}}"'] \arrow[d, "\mathcal{C}_{\mathrm{ev}}^{\flat,\{1,2\}}"'] & \\
\mathbb{Q}_\ell\Big|_{\overline{\eta}} \otimes \mathcal{H}^{j}_{\{3\},W}\Big|_{\overline{v}} \arrow[r, "\mathfrak{sp}^*"] & \mathbb{Q}_\ell\Big|_{\overline{\eta}} \otimes \mathcal{H}^{j}_{\{3\},W}\Big|_{\overline{\eta}}
\end{tikzcd}$$

By "Zorro" lemma, the composition of the right vertical morphisms is the identity. Thus $\mathfrak{sp}^*$ is surjective.

Some general remarks

1. When there is level structure $N \subset X$, the cohomology sheaf $\mathcal{H}^j_{G,N,I,W}$ is ind-smooth over $(X \smallsetminus N)^I$.

2. The same argument works for any reductive group over F.

3. The same argument works for cohomology with $\mathbb{Z}_\ell$-coefficients.

4. an application of the smoothness property : see [Arinkin-Gaitsgory-Kazhdan-Raskin-Rozenblyum-Varshavsky]