Math 123, Practice Exam #2 Solutions, November 10, 2000

1. Find the following:

   (a) This is an application of L'Hopital's rule.

$$\lim_{y\to\infty}\left(1+\frac{2}{y}\right)^y = \lim_{y\to\infty}\exp\left(y\ln\left(1+\frac{2}{y}\right)\right) = \exp\left(\lim_{y\to\infty}\frac{\ln\left(1+\frac{2}{y}\right)}{\frac{1}{y}}\right) =$$

$$\exp\left(\lim_{y\to\infty}\frac{\frac{1}{1+\frac{2}{y}}\left(-\frac{2}{y^2}\right)}{-\frac{1}{y^2}}\right) = \exp(2) = e^2$$

   (b) This is an application of L'Hopital's rule, twice.

$$\lim_{x\to 0}\frac{\cos\left(\sqrt{5}\,x\right)-1}{x^2} = \lim_{x\to 0}\frac{-\sqrt{5}\sin\left(\sqrt{5}\,x\right)}{2x} = \lim_{x\to 0}\frac{-5\cos\left(\sqrt{5}\,x\right)}{2} = -\frac{5}{2}$$

   (c) This is an indeterminate form of type $\infty-\infty$.

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = \lim_{x\to 0}\frac{\sin x - x}{x\sin x} = \lim_{x\to 0}\frac{\cos x - 1}{x\cos x+\sin x} = \lim_{x\to 0}\frac{-\sin x}{\cos x - x\sin x+\cos x} = 0$$

   (d) The horizontal and vertical asymptotes of

$$y = \frac{4-3x}{\sqrt{16x^2+1}}$$

   Since $\lim_{x\to\infty}\frac{4-3x}{\sqrt{16x^2+1}} = -\frac{3}{4}$ and $\lim_{x\to-\infty}\frac{4-3x}{\sqrt{16x^2+1}} = \frac{3}{4}$, the horizontal asymptotes are $y=\frac{3}{4}$ and $y=-\frac{3}{4}$. There are no vertical asymptotes since $\frac{4-3x}{\sqrt{16x^2+1}}$ exists for all x values.

   (e) $f'(x)$ where

$$f(x) = \frac{\ln x}{x}$$

   Use the quotient rule to obtain

$$f'(x) = \frac{x\frac{1}{x}-\ln x}{x^2} = \frac{1-\ln x}{x^2}$$

   (f) $f'(x)$ where

$$f(x) = x^{x^2}$$

   First, rewrite as $f(x) = \exp\left(x^2\ln x\right)$ then

$$f'(x) = \exp\left(x^2\ln x\right)\left(2x\ln x + x^2\frac{1}{x}\right) = x^{x^2}\left(2x\ln x + x\right)$$

(g) $f'(x)$ where

$$f(x) = \arctan(x^3)$$

$$f'(x) = \frac{1}{1 + (x^3)^2}\, 3x^2 = \frac{3x^2}{1 + x^6}$$

2. Find the equation for the tangent line to the curve given by the equation $y^2 = x + \cos(xy)$ through the point $(0, -1)$.

   Differentiating both sides of the equation,

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x + \cos(xy))$$

   yields

$$2y\frac{dy}{dx} = 1 + (x\frac{dy}{dx} + y)(-\sin(xy)).$$

   Solving for $\frac{dy}{dx}$, one obtains

$$\frac{dy}{dx} = \frac{1 - y\,\sin(x\,y)}{2\,y + x\,\sin(x\,y)}.$$

   Plugging in $(x, y) = (0, -1)$, one obtains $\frac{dy}{dx} = -\frac{1}{2}$. The equation of the tangent line thru $(0, -1)$, then, is given by

$$y - (-1) = -\frac{1}{2}(x - 0)$$

   or

$$y = -1 - \frac{1}{2}x.$$

3. A man 6 feet tall is walking away from a light pole which is 30 feet high. If the tip of his shadow is moving at a rate equal to the distance between him and the light pole (in feet) then how fast is the man walking when he is 24 feet from the pole?

   Let l be the distance from the man to the lamp pole, x the distance from the lamp pole to the tip of the man's shadow, and t be time. We are told that $\frac{dx}{dt} = l$. We want to know what $\frac{dl}{dt}$ is when $l = 24$. By similar triangles, we obtain

$$\frac{6}{30} = \frac{x - l}{x}$$

   which can be solved to obtain $l = \frac{4}{5}x$. Therefore,

$$\frac{dl}{dt} = \frac{d}{dt}(\frac{4}{5}x) = \frac{4}{5}\frac{dx}{dt} = \frac{4}{5}l$$

   so when $l = 24$, $\frac{dl}{dt} = \frac{96}{5}$.

4. A spherical snowball is melting at a rate equal to its surface area. How fast is its radius shrinking when its volume is equal to its surface area?

   Let r be the radius of the spherical snowball, V the volume of the sphere, A be the surface area of the snowball, and t be time. We are told that $\frac{dV}{dt} = -A$. We want to know what $-\frac{dr}{dt}$ is when $V = A$. Recall that $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$ so

$$\frac{dV}{dt} = \frac{d}{dt}(\frac{4}{3}\pi r^3) = 4\pi r^2 \frac{dr}{dt} = A\frac{dr}{dt}.$$

   On the other hand, $\frac{dV}{dt} = -A$, therefore, combining with the previous equation, $\frac{dr}{dt} = -1$ at any moment in time.

5. Two nonnegative numbers are such that the sum of the first number and 3 times the second number equals 10. Find these numbers if the sum of their squares is as small as possible.

   Call the numbers x and y. We know that $x + 3y = 10$ or, in other words, $x = 10 - 3y$. Furthermore, let $f = x^2 + y^2 = (10 - 3y)^2 + y^2$. Notice that the fact that both x and y be nonnegative means that y must lie in the interval $[0, \frac{10}{3}]$. We want to find the absolute minimum of $f(y)$ on the interval $[0, \frac{10}{3}]$.

   Since $f'(y) = -60 + 20y$, the only stationary point of f (i.e. where $f'(y) = 0$) is $y = 3$. There are no singular points of f. Therefore, the only critical point of f is 3. The global minimum of f is then the smallest value of f evaluated on the endpoints (namely $f(0) = 100$, $f(\frac{10}{3}) = \frac{100}{9}$) or $f(3) = 10$. Therefore, the two numbers are $x = 10 - 3(3) = 1$ and $y = 3$.

6. Consider the functon

$$f(x) = 3x^4 - 4x^3 + 20000$$

   (a) On what interval(s) is f increasing? Since

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$$

   $f'(x)$ is positive on the interval $(1, \infty)$.

   (b) On what interval(s) is f concave down? Since

$$f''(x) = -24x + 36x^2 = 12x(-2 + 3x)$$

   $f''(x)$ is negative on the interval $(0, \frac{2}{3})$.

   (c) Find the inflection point(s) of f.

   The concavity changes across $x = \frac{2}{3}$ and $x = 0$.

   (d) Find the critical points of f.

   The stationary points are $x = 0, 1$. There are no singular points.

   (e) Find the local maximum (maxima) of f.

   There are no local maxima by the first derivative test.

(f) Find the global minimum of f on the interval $[-2, 3]$.

$x = 1$ is a local minimum by the first derivative test. Evaluating f on the endpoints ($f(-2) = 20080$, $f(3) = 20135$) and $f(1) = 19999$, the smallest is 19999 which occurs at $x = 1$.

7. Suppose the graph on the following page is of $y = f'(x)$ (**NOT** $f(x)$).

(a) Find the critical numbers of f.

The stationary points are where $f'(x) = 0$ which occurs when $x = -4, 0, 4$. The singular points are where $f'(x)$ does not exist. There are no such points in this case. Therefore, $x = -4, 0, 4$ are the critical numbers.

(b) On what interval(s) is f increasing?

$f(x)$ is increasing on the intervals $(-4, 0)$ and $(0, 4)$.

(c) On what interval(s) is f concave down?

$f(x)$ is concave down on $(-2, 0)$ and $(2, 6)$.

(d) Find the values of x on the interval $(-\infty, \infty)$ where f has a local minimum.

$f(x)$ has a local minimum when $x = -4$ only.

(e) Find the values of x on the interval $[0, 4]$ where f has a global minimum.

$x = 0$ is the global minimum since $f(x)$ is increasing on $(0, 4)$.

(f) Find the x values of all inflection points of f.

$x = -2, 0, 2$