Math 123, Practice Exam #3 Solutions, December 6, 2000

1. Find the following:

(a)

$$\int (5t^3 - \frac{7}{t^4} + 6\sqrt{t} - 4\sin t)\, dt = \int (5t^3 - 7t^{-4} + 6t^{\frac{1}{2}} - 4\sin t)\, dt$$
$$= \frac{5}{4}t^4 + \frac{7}{3}t^{-3} + 4t^{\frac{3}{2}} + 4\cos t + C$$

(b)

$$\int (2e^x + \frac{3}{x} - \pi^x + 7\sec^2 x)\, dx = 2e^x + 3\ln|x| - \frac{\pi^x}{\ln \pi} + 7\tan x + C$$

(c)

$$\int_1^2 \frac{1-x^3}{x^2}\, dx = \int_1^2 \left(\frac{1}{x^2} - x\right) dx = \left(-x^{-1} - \frac{x^2}{2}\right)\Big|_1^2 = -1$$

(d)

$$\frac{d}{dx}\int_3^x \sqrt{t^3+1}\, dt = \sqrt{x^3+1}$$

by using the fundamental theorem of calculus.

(e)

$$\int_0^{\sqrt{\pi}} \frac{d}{dt}\cos t^2\, dt = \cos t^2\Big|_0^{\sqrt{\pi}} = \cos\pi - \cos 0 = -2$$

(f)

$$\int_{-1}^{10} f(x)\, dx = \int_{-1}^{0} f(x)\, dx + \int_0^3 f(x)\, dx + \int_3^{10} f(x)\, dx$$
$$= \int_{-1}^{0} (-x^2)\, dx + \int_0^3 (2x)\, dx + \int_3^{10} (-5)\, dx = -\frac{1}{3} + 9 - 35 = -\frac{79}{3}.$$

2. Michelle begins walking along a line at time $t = 0$. Her acceleration at time $t \geq 0$ is $a(t) = 6t - 7$. Suppose that her initial velocity is 1 and her initial position is 3. If $s(t)$ denotes her position at time t and $v(t)$ denotes her velocity at time t then anser the following:

(a) Find her velocity at $t = 4$. Since $v(t)$ is an antiderivative of $a(t)$,

$$v(t) = \int (6t - 7)\, dt = 3t^2 - 7t + v_0$$

where v_0 is a constant. Furthermore, $1 = v(0) = v_0$ so

$$v(t) = \int (6t - 7)\, dt = 3t^2 - 7t + 1.$$

Therefore, $v(4) = 3(4)^2 - 7(4) + 1 = 21$.

(b) Find her position at $t = 2$. Since $s(t)$ is an antiderivative of $v(t)$,

$$s(t) = \int (3t^2 - 7t + 1)\, dt = t^3 - \frac{7}{2}t^2 + t + s_0$$

where s_0 is a constant. Furthermore, $3 = s(0) = s_0$ so

$$s(t) = t^3 - \frac{7}{2}t^2 + t + 3.$$

Therefore, $s(2) = 2^3 - \frac{7}{2}(2)^2 + 2 + 3 = -1$.

(c) When does she return to her starting position? This occurs when $s(t) = s(0)$ where $t > 0$. But $s(0) = 3$ so we are really solving the equation:

$$t^3 - \frac{7}{2}t^2 + t = 0$$

The left hand side is

$$t(t^2 - \frac{7}{2}t + 1) = 0.$$

Now, since we are interested in $t > 0$, we need to solve:

$$t^2 - \frac{7}{2}t + 1 = 0$$

By the quadratic formula, Michelle returns to the starting point when

$$t = \frac{7 \pm \sqrt{33}}{4}$$

both of which are positive.

3. A box with a square base and open top has a total surface area of 300 square centimeters. Find the dimensions of the box which maximize its volume.

Let x be the length of a side of the square base and y be the height of the box. The surface area of the box is

$$x^2 + 4xy = 300$$

so we can solve for y to obtain

$$y = \frac{300 - x^2}{4x} = \frac{75}{x} - \frac{x}{4}.$$

On the other hand, the volume V of the box is given by

$$V = x^2y = x^2(\frac{75}{x} - \frac{x}{4})$$

so

$$V(x) = 75x - \frac{x^3}{4}.$$

We want to find the maximum of $V(x)$ where $0 < x < \infty$.

$$V'(x) = 75 - \frac{3}{4}x^2 = 0$$

when $x = 10$ (we rule out $x = -10$ since $x > 0$). On the other hand, $V'(x) > 0$ when $0 < x < 10$ and $V'(x) < 0$ when $x > 10$ so $V(10)$ must be the maximum.

When $x = 10$ then $y = \frac{75}{10} - \frac{10}{4} = 5$, thus, the box which maximizes volume has a 10 x 10 base and a height of 5.

4. Consider the following Riemann sum:

$$I := \lim_{n \to \infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^8 \left(\frac{2}{n}\right).$$

(a) Write I as a definite integral.

$$\int_1^3 x^8 \, dx$$

(b) Calculate I (using any method you like).

$$\int_1^3 x^8 \, dx = \frac{3^9}{9} - \frac{1^9}{9} = \frac{19682}{9}$$

5. Consider the graph below. Find the following:

(a)

$$\int_{-5}^{0} f(x)\,dx = \int_{-5}^{-4} f(x)\,dx + \int_{-4}^{-3} f(x)\,dx + \int_{-3}^{0} f(x)\,dx = (3)(1) + (-4)(1) - \frac{1}{2}(4)(3) = -7.$$

(b)

$$F'(x) = \frac{d}{dx}\int_{-6}^{x} f(t)\,dt = f(x)$$

Therefore, $F'(-3) = f(-3) = -4$.

(c)

$$\int_{-3}^{1} |\, f(x) \,|\, dx = \int_{-3}^{0} |\, f(x) \,|\, dx + \int_{0}^{1} |\, f(x) \,|\, dx = \frac{1}{2}(3)(4) + \frac{1}{2}(1)(2) = 7$$

(d)

$$\int_{-3}^{4} f'(x)\,dx = f(4) - f(-3) = -1 - (-4) = 3.$$

(e) Let $u = x^2$ then $du = \frac{du}{dx}\,dx = 2xdx$. Furthermore, $(-1)^2 = 1$ and $2^2 = 4$ thus,

$$\int_{-1}^{2} f'(x^2)\,x\,dx = \int_{1}^{4} f'(u)\,\frac{1}{2}\,du = \frac{1}{2}(f(4) - f(1)) = \frac{1}{2}(-1-2) = -\frac{3}{2}.$$

(f)

$$\int_{5}^{7} (9(f(x))^2 - 8)\,dx = \int_{5}^{7} (9(2)^2 - 8)\,dx = 56$$