Math 563A1, Midterm Exam, October 16, 2000
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This exam is due at the beginning of class on Monday October 23. The exam is open book and you may work with others although you must write up the solutions yourself. Good luck!

1. (10 points) Exercise 1.5.4 from our text.

2. (10 points) Consider the map $\alpha : \mathbf{R} \to \mathbf{R}^3$ given by

$$\alpha(t) := \begin{cases} (t^2, e^{-\frac{1}{t^2}}, 2), & \text{if } t \neq 0; \\ (0,0,2) & \text{if } t = 0 \end{cases}.$$

   (a) For what values of t is $\alpha(t)$ a differentiable curve?

   (b) For what values of t is $\alpha(t)$ a regular curve?

   (c) Calculate its Frenet frame, curvature and torsion.

3. (10 points) Let $\alpha : I \to \mathbf{R}$ be a regular curve parametrized by arc length whose curvature is nowhere vanishing. Let

$$x(s,v) = \alpha(s) + r\,(n(s)\cos(v) + b(s)\,\sin(v))$$

   (where r is a nonzero constant and s belongs to I) be a parametrized surface (the *tube* of radius r around α), where n is the normal vector and b is the binormal vector of α. Show that if x is regular then its unit normal vector is

$$N(s,v) = -(n(s)\cos(v) + b(s)\sin(v)).$$

4. (10 points) Let Σ be a surface parametrized by a coordinate chart x. A *parallel surface* to x is a surface given by the coordinate chart y where

$$y(u,v) := x(u,v) + aU(u,v)$$

   where U is a unit normal to Σ and a is a constant.

   (a) Prove that $y_u \times y_v = (1 - 2Ha + Ka^2)(x_u \times x_v)$ where K and H are the Gaussian and Mean curvatures, respectively.

   (b) Prove that at the regular points, the Gaussian curvature of y is

$$\frac{K}{1 - 2Ha + Ka^2}$$

   and the Mean curvature of y is

$$\frac{H - Ka}{1 - 2Ha + ka^2}.$$

   (c) Let a surface x have constant mean curvature equal to $c \neq 0$ and consider the parallel surface to Σ at a distance $1/2c$. Prove that the parallel surface has constant Gaussian curvature equal to $4c^2$.

5. (10 points) Determine the umbilical points of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$