Math 721A1, Homework #2
Differential Topology I

1. Let V be a vector space with a bilinear operation $V \times V \to V$ denoted by $(X, Y) \mapsto [X, Y]$ satisfying the conditions
$$[X, Y] = -[Y, X]$$
and
$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$
for all X, Y, and Z in V. The pair $(V, [\cdot, \cdot])$ is called a *Lie algebra*.
Let X and Y be vector fields on a smooth manifold M. Define $[X, Y]$ to be the map $C^\infty(M) \to C^\infty(M)$ defined via
$$[X, Y](f) := X(Y(f)) - Y(X(f))$$
for all f in $C^\infty(M)$. Prove that this makes the set of vector fields on M into a Lie algebra.
2. Let $\pi : E \to B$ be a smooth, rank k vector bundle and $\pi' : E' \to B$ be a smooth, rank k' vector bundle. Let $E'' := \{(e, e') \,|\, e \in E, e' \in E', \pi(e) = \pi'(e')\}$ and let $\pi'' : E'' \to B$ be defined by $\pi''((e, e')) := \pi(e)$.
   (a) Show that $\pi'' : E'' \to B$ is a smooth, rank $(k + k')$ bundle whose fiber E''_b is isomorphic to the direct sum $E_b \oplus E'_b$ for all b in B.
   This bundle $\pi'' : E'' \to B$ is often denoted by $\pi : E \oplus E' \to B$ and is called the *direct (or Whitney) sum of the bundles* $\pi : E \to B$ and $\pi' : E' \to B$.
   (b) If $\pi : E \to B$ is an orientable vector bundle and $\pi' : E' \to B$ is a nonorientable vector bundle then prove that $\pi'' : E \oplus E' \to B$ is a nonorientable vector bundle.
3. Let M and N be smooth manifolds. Prove that if M is orientable and N is orientable then $M \times N$ is orientable.
4. Let M be a smooth, m-dimensional manifold and N a smooth, n-dimensional manifold and $f : M \to N$ be a smooth map. Let (x, U) and (y, V) be charts around p and $f(p)$, respectively.
   (a) If $g : N \to \mathbf{R}$ then
   $$\frac{\partial(g \circ f)}{\partial x^i}(p) = \sum_{j=1}^{n} \frac{\partial g}{\partial y^j}(f(p)) \frac{\partial y^j \circ f}{\partial x^i}(p).$$
   (b) Show that
   $$f_*\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \sum_{j=1}^{n} \frac{\partial(y^j \circ f)}{\partial x^i}(p) \, \frac{\partial}{\partial y^j}\bigg|_{f(p)}.$$
   More generally, express
   $$f_*\left(\sum_{k=1}^{m} a^k \frac{\partial}{\partial x^k}\bigg|_p\right)$$
   in terms of the
   $$\frac{\partial}{\partial y^j}\bigg|_{f(p)}.$$

(c) Show that

$$(f^* dy^j)(p) = \sum_{i=1}^{m} \frac{\partial y^j \circ f}{\partial x^i}(p) dx^i(p).$$

(d) Express

$$f^* \left(\sum_{j_1,\cdots,j_k} a_{j_1,\cdots,j_k} dy^{j_1} \otimes \cdots \otimes dy^{j_k} \right)$$

in terms of the dy^j.

5. Show that if $f : M \to \mathbf{R}$ is a smooth map then for all v in T_pM,

$$f_*(v) = df(v)_{f(p)} \in T_{f(p)}\mathbf{R}.$$

6. Let f, g in $C^\infty(M)$. Show that

$$d(fg) = gdf + fdg.$$

7. Consider the Euclidean metric on $\mathbf{R}^2$ in Cartesian coordinates $x := (x^1, x^2)$ given by

$$g := \sum_{i=1}^{2} dx^i \otimes dx^i.$$

Let (r, θ) be polar coordinates where $x = r\cos\theta$ and $y = r\sin\theta$ and where $(r, \theta) \in (0, \infty) \times (0, 2\pi)$. Write g in these coordinates.