Math 721A1, Homework #3
Differential Topology I

Throughout, a *tensor of type* $\binom{k}{l}$ *on a smooth manifold* M is a section of the vector bundle$(T^*M)^{\otimes k} \otimes (TM)^{\otimes l} \to M$.

1. Let M be a manifold with a Riemannian metric g.
   (a) Define the $\binom{1}{1}$ tensor $g^\flat : TM \to T^*M$ via
   $$g^\flat(X)(Y) := g(X, Y)$$
   for all X and Y in T_pM for all p in M. Prove that $g^\flat$ restricted to each fiber is a linear isomorphism. What kind of map is $g^\flat$?
2. Prove the following.
   (a) If $\alpha : M \to N$ is smooth then $\alpha_* : TM \to TN$ is smooth.
   (b) If $\alpha : M \to N$ is a diffeomorphism and X is a smooth vector field on M then $\alpha_*(X)$ is a smooth vector field on N.
   (c) If $\alpha : \mathbb{R} \to \mathbb{R}$ is $\alpha(t) := t^3$ then there is a smooth vector field X on $\mathbb{R}$ such that $\alpha_*(X)$ is not a smooth vector field.
3. On $\mathbb{R}^3$, let X, Y, Z be vector fields
   $$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$
   $$Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z},$$
   $$Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$
   (a) Show that the map
   $$aX + bY + cZ \mapsto (a, b, c) \in \mathbb{R}^3$$
   is an isomorphism (from a certain set of vector fields to $\mathbb{R}^3$) and that $[U, V] \mapsto$ the cross product of the images of U and V.
   (b) Show that the flow of $aX + bY + cZ$ is a rotation of $\mathbb{R}^3$ about some axis through 0.
4. If A is a tensor field of type $\binom{k}{l}$ on N and $\phi : M \to N$ is a diffeomorphism, we defined $\phi^* A$ on M as follows. If $v_1, \ldots, v_k$ in T_pM and $\lambda_1, \ldots, \lambda_l$ in T_p^*M then define
   $$(\phi^*A)(p)(v_1, \ldots, v_k, \lambda_1, \ldots, \lambda_l) := A(\phi(p))(\phi_* v_1, \ldots, \phi_* v_k, (\phi^{-1})^* \lambda_1, \ldots, (\phi^{-1})^* \lambda_l).$$
   (a) Check that under our identification of a vector field (or covector field) with a tensor field of type $\binom{0}{1}$ (or type $\binom{1}{0}$), this agrees with our old $\phi^* Y$ ($\phi^* \omega$).
   (b) If X is a vector field on M which generates $\{\phi_t\}$ and A is a tensor field of type $\binom{k}{l}$ on M, we define
   $$(L_X A)(p) := \lim_{h \to 0} \frac{(\phi_h^* A)(p) - A(p)}{h}.$$
   Show that
   $$L_X(A + B) = L_X A + L_X B,$$
   $$L_X(A \otimes B) = (L_X A) \otimes B + A \otimes (L_X B),$$

and

$$L_{X_1+X_2}\, A \;=\; L_{X_1}\, A \;+\; L_{X_2}\, A$$

(in particular, $L_X\,(f\,A) \;=\; X(f)\,A \;+\; f L_X\,A$ for all smooth functions f on M).

(c) If ω is a tensor field of type $\binom{1}{0}$ on M then show that

$$L_X(\omega(Y)) \;=\; (L_X\omega)(Y) \;+\; \omega(L_X Y).$$

(d) Let g be a tensor field of type $\binom{2}{0}$ tensor on M, e.g. g could be a Riemannian metric on M. Show that

$$(L_X\, g)(X_1, X_2) \;=\; L_X\,(g(X_1, X_2)) - g(L_X\, X_1, X_2) + g(X_1\, L_X, X_2).$$