Math 721A1, Homework #4
Differential Topology I

1. Show that n functions $f_1, \ldots, f_n : M \to \mathbb{R}$ form a coordinate system in a neighborhood of p in M if and only if
$$(df_1 \wedge df_2 \cdots \wedge df_n)_p \neq 0.$$

2. Let $v_1, \ldots, v_n$ be a basis for V and $v_1^* \ldots, v_n^*$ be its dual basis, and $w_1, \ldots, w_n$ be another basis for V with its dual basis $w_1^*, \ldots, w_n^*$ such that
$$w_i = \sum_{j=1}^{n} \alpha_{ji} v_j$$
for all $i = 1, \ldots, n$. Show that
$$\det(\alpha_{ij})\, w_1^* \wedge \cdots \wedge w_n^* = v_1^* \wedge \cdots \wedge v_n^*.$$

3. If ω is a $(k+1)$-form ($k \geq 0$) then $\iota_X \omega$ is the k-form defined by
$$(\iota_X \omega)(X_1, \ldots, X_k) := \omega(X, X_1, \ldots, X_k).$$
$\iota_X \omega$ is called the interior product (or contraction) of ω with X. (Define $\iota_X f := 0$ for any smooth function f.) Show the following.
(a) Show that for any X, k-form ω_1, and l-form ω_2,
$$\iota_X(\omega_1 \wedge \omega_2) = (\iota_X \omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (\iota_X \omega_2)$$
(b) Show that
$$L_X(\omega_1 \wedge \omega_2) = (L_X \omega_1) \wedge \omega_2 + \omega_1 \wedge (L_X \omega_2)$$
(c) Show that
$$L_X \omega = d(\iota_X \omega) + \iota_X(d\omega).$$
(d) Using the previous, show that
$$d(L_X \omega) = L_X(d\omega).$$

4. Consider $\mathbb{R}^3$ with the standard Euclidean metric so that $T_p\mathbb{R}^3$ is identified with $T_p^*\mathbb{R}^3$. Let $X = \sum_{i=1}^3 a^i \frac{\partial}{\partial x^i}$ be a vector field on $\mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ a function. Define the gradient of f by
$$\operatorname{grad} f := \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i},$$
the divergence of X by
$$\operatorname{div} X := \sum_{i=1}^{n} \frac{\partial a^i}{\partial x^i},$$
and the curl of X by
$$\operatorname{curl} X := \left(\frac{\partial a^3}{\partial x^2} - \frac{\partial a^2}{\partial x^3}\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial a^1}{\partial x^3} - \frac{\partial a^3}{\partial x^1}\right)\frac{\partial}{\partial x^2} + \left(\frac{\partial a^2}{\partial x^1} - \frac{\partial a^1}{\partial x^2}\right)\frac{\partial}{\partial x^3}.$$
Define the differential forms on $\mathbb{R}^3$,
$$\omega_X = a^1\, dx^1 + a^2\, dx^2 + a^3\, dx^3$$

and

$$\eta_X = a^1\, dx^2 \wedge dx^3 + a^2\, dx^3 \wedge dx^1 + a^3\, dx^1 \wedge dx^2.$$

(a) Show that

$$df = \omega_{\mathrm{grad} f},$$

$$d(\omega_X) = \eta_{\mathrm{curl} X}$$

$$d(\eta_X) = (\mathrm{div} X)\, dx^1 \wedge dx^2 \wedge dx^3.$$

(b) Conclude that

$$\mathrm{curl}\,(\mathrm{grad}\, f) = 0$$

and

$$\mathrm{div}\,(\mathrm{curl}\, X) = 0.$$

5. Prove that any smooth manifold M has a Riemannian metric. *Hint:* Use the existence of partitions of unity and the existence of the standard dot product on open subsets of $\mathbb{R}^n$ to construct a Riemannian metric on M.