Math 123, Practice Exam Solutions for Exam #1, October 13, 2003

1. Calculate the following:

   (a)
   $$\lim_{x\to-1}\frac{x^2-4x-5}{x+1}=\lim_{x\to-1}\frac{(-5+x)(1+x)}{x+1}=\lim_{x\to-1}(-5+x)=-6$$

   (b)
   $$\lim_{x\to3}\frac{5x^2}{2x-1}=\frac{5(3)^2}{2(3)-1}=9$$

   (c)
   $$\lim_{x\to2^+}\frac{4-x^2}{|2-x|}=\lim_{x\to2^+}\frac{4-x^2}{-(2-x)}=\lim_{x\to2^+}\frac{(2-x)(x+2)}{-(2-x)}=\lim_{x\to2^+}-(x+2)=-4$$

   (d)
   $$\lim_{x\to1}\sqrt{\frac{2x^3-3x+5}{2-x}}=\sqrt{\frac{2\cdot1^3-3\cdot1+5}{2-1}}=\sqrt{4}=2$$

   (e)
   $$\lim_{x\to3}f(x)\ \ \text{where}\ \ f(x)=\begin{cases}x^2 & \text{if } x>3\\ 8 & \text{if } x=3\\ 12-x & \text{if } x<3\end{cases}$$

   Notice that $\lim_{x\to3^-}f(x)=\lim_{x\to3^-}(12-x)=9$ and $\lim_{x\to3^+}f(x)=\lim_{x\to3^+}x^2=9$. Since both one-sided limits agree with the value 9, $\lim_{x\to3}f(x)=9$.

   (f)
   $$\lim_{x\to3^-}\frac{x+3}{x^2-9}=\lim_{x\to3^-}\frac{x+3}{(x-3)(x+3)}=\lim_{x\to3^-}\frac{1}{x-3}=-\infty$$

   (g) The horizontal and vertical asymptotes of
   $$y=\frac{4-3x}{\sqrt{16x^2+1}}$$

   Since $\lim_{x\to\infty}\frac{4-3x}{\sqrt{16x^2+1}}=-\frac{3}{4}$ and $\lim_{x\to-\infty}\frac{4-3x}{\sqrt{16x^2+1}}=\frac{3}{4}$, the horizontal asymptotes are $y=\frac{3}{4}$ and $y=-\frac{3}{4}$. There are no vertical asymptotes since $\frac{4-3x}{\sqrt{16x^2+1}}$ exists for all x values.

   (h) $f'(x)$ where
   $$f(x)=\sin\left(x^{100}\right)$$

   Applying the chain rule, we obtain
   $$f'(x)=100x^{99}\cos\left(x^{100}\right)$$

(i) $f'(x)$ where
$$f(x) = \sqrt{e^{2x} + 7x}$$

Applying the chain rule, we obtain
$$f'(x) = \frac{1}{2}(e^{2x} + 7x)^{-\frac{1}{2}}(2e^{2x} + 7).$$

(j) $f'(x)$ where
$$f(x) = 10^{\cos x}$$

Rewrite $10^{\cos x} = e^{\cos x \ln 10}$ then applying the chain rule, we obtain
$$f'(x) = e^{\cos x \ln 10}(-\sin x \ln 10) = -(\sin x \ln 10)10^{\cos x}.$$

(k) $f'(x)$ where
$$f(x) = \frac{\ln x}{x}$$

Use the quotient rule to obtain
$$f'(x) = \frac{x\frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

(l) $f'(x)$ where
$$f(x) = x^{x^2}$$

First, rewrite as $f(x) = \exp(x^2 \ln x)$ then
$$f'(x) = \exp(x^2 \ln x)\left(2x \ln x + x^2\frac{1}{x}\right) = x^{x^2}(2x \ln x + x)$$

(m) $f'(x)$ where
$$f(x) = \arctan(x^3)$$
$$f'(x) = \frac{1}{1 + (x^3)^2} 3x^2 = \frac{3x^2}{1 + x^6}$$

2. Consider the function
$$f(x) = \begin{cases} x - c, & \text{if } x > 2; \\ 3x^2, & \text{if } x \le 2 \end{cases}$$
where c is a real number.

   (a) What value of c makes the function f continuous everywhere? The only place where the function f might not be continuous is at $x = 2$ but we can avoid this if we choose c so that the graphs of $y = x - c$ and $y = 3x^2$ agree at $x = 2$, *i.e.* so there is no jump across $x = 2$. Solving $2 - c = 3 \cdot 2^2$ means that $c = -10$.

   (b) If $x > 2$ then $f(x) = x - c \Rightarrow f'(x) = 1 \Rightarrow f'(7) = 1$

   (c) If $x < 2$ then $f(x) = 3x^2 \Rightarrow f'(x) = 6x \Rightarrow f'(-1) = -6$

(d) Call $f_1(x) = x - c$ and $f_2(x) = 3x^2$. Notice that $f'(2)$ does not exist (even when $c = -10$) since $f_1'(2) = 1$ which does not agree with $f_2'(2) = 6 \cdot 2 = 12$.

3. Compute the derivative of the following functions:

(a)
$$f(x) = \pi^4 \Rightarrow f'(x) = 0$$

(b)
$$f(x) = 3x^5 - x^2 + 9 \Rightarrow f'(x) = 15x^4 - 2x$$

(c)
$$f(x) = \frac{2}{x^2} - 3\sqrt{x} = 2x^{-2} - 3x^{\frac{1}{2}} \Rightarrow f'(x) = -4x^{-3} - \frac{3}{2}x^{-\frac{1}{2}}$$

(d)
$$f(x) = \frac{x^2}{2x - 3} \Rightarrow$$

$$f'(x) = \frac{(2x - 3)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(2x - 3)}{(2x - 3)^2} = \frac{(2x - 3)(2x) - x^2(2)}{(2x - 3)^2} = \frac{2x^2 - 6x}{(2x - 3)^2}$$

(e)
$$f(x) = x^2 e^x \Rightarrow f'(x) = e^x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(e^x) = 2xe^x + x^2 e^x$$

(f)
$$f(t) = t\sin t \Rightarrow f'(t) = t\frac{d}{dt}(\sin t) + \sin t\frac{d}{dt}(t) = t\cos t + \sin t$$

4. Find the equation for the tangent line to the curve $y = f(x)$ where through the point $(1, -3)$ where
$$f(x) = x^8 - 4x.$$

$f'(x) = 8x^7 - 4$. The slope of the tangent line is $f'(1) = 4$. Therefore the equation for this tangent line is

$$\frac{y - (-3)}{x - 1} = 4$$

or, solving for y, $y = 4x - 7$.

5. Suppose an object is moving along the real line with its position at time t given by the function $s(t) = \frac{1}{3}t^3 - 3t^2 - 7t + 10$.

(a) The object is at rest when the velocity $v(t) = 0$ but $v(t) = s'(t) = t^2 - 6t - 7 = (t - 7)(t + 1) = 0$. The solution is $t = -1$ or $t = 7$.

(b) The object is decelerating when the acceleration $a(t) < 0$ but $a(t) = v'(t) = 2t - 6 < 0$ when $t < 3$.

(c) $v(2) = -15$.

6. Find the equation for the tangent line to the curve given by the equation $\cos(xy) - 3y^3 = e^x + 1$ through the point $(0, -1)$.

   We need to find $\frac{dy}{dx}$. Use implicit differentiation to obtain

   $$\frac{d}{dx}(\cos(xy) - 3y^3) = \frac{d}{dx}(e^x + 1)$$

   which yields

   $$-\sin(xy)(x\frac{dy}{dx} + y) - 9y^2\frac{dy}{dx} = e^x$$

   Now, solve to obtain

   $$\frac{dy}{dx} = -\frac{e^x + y\sin(xy)}{9y^2 + x\sin(xy)}$$

   At the point $(x, y) = (0, -1)$, $\frac{dy}{dx} = -\frac{1}{9}$ so the equation for the tangent line is

   $$\frac{y - (-1)}{x - 0} = -\frac{1}{9}$$

   or $y = -1 - \frac{1}{9}x$.

7. Consider the graph of $y = f(x)$ on the next page (figure 1).

   (a) Where is f undefined?
   $$x = -1, 0, 3$$

   (b) Where is f not continuous?
   $$x = -5, -1, 0, 3$$

   (c) Where is f not differentiable?
   $$x = -5, -3, -2, -1, 0, 3, 5$$

   (d) On what interval(s) is f' positive? Where does f' vanish? f' is positive on the intervals $(-5, -4)$, $(-2, -1)$, $(2, 3)$, $(3, 5)$. f' vanishes at $x = -4, 2$

   (e) On what interval(s) is f concave down? $(-\infty, -5), (-5, -3), (-1, 0), (3, 5)$.

   (f) What are
   i.
   $$f'(6) = \frac{0 - 3}{7.25 - 5} = -\frac{3}{2.25}$$
   ii.
   $$\lim_{x \to -5+} f(x) = -2$$
   iii.
   $$\lim_{x \to -1} f(x) = -1$$
   iv.
   $$\lim_{x \to -\infty} f(x) = 0$$