Math 123, Practice Exam #2 Solutions, November 10, 2003

1. Find the following:

   (a) This is an application of L'Hopital's rule.

$$\lim_{y\to\infty}\left(1+\frac{2}{y}\right)^y = \lim_{y\to\infty}\exp\left(y\ln\left(1+\frac{2}{y}\right)\right) = \exp\left(\lim_{y\to\infty}\frac{\ln\left(1+\frac{2}{y}\right)}{\frac{1}{y}}\right) =$$

$$\exp\left(\lim_{y\to\infty}\frac{\frac{1}{1+\frac{2}{y}}\left(-\frac{2}{y^2}\right)}{-\frac{1}{y^2}}\right) = \exp(2) = e^2$$

   (b) This is an application of L'Hopital's rule, twice.

$$\lim_{x\to 0}\frac{\cos\left(\sqrt{5}\,x\right)-1}{x^2} = \lim_{x\to 0}\frac{-\sqrt{5}\sin\left(\sqrt{5}\,x\right)}{2x} = \lim_{x\to 0}\frac{-5\cos\left(\sqrt{5}\,x\right)}{2} = -\frac{5}{2}$$

   (c) This is an indeterminate form of type $\infty-\infty$.

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = \lim_{x\to 0}\frac{\sin x - x}{x\sin x} = \lim_{x\to 0}\frac{\cos x - 1}{x\cos x+\sin x} = \lim_{x\to 0}\frac{-\sin x}{\cos x - x\sin x+\cos x} = 0$$

   (d) The horizontal and vertical asymptotes of

$$y = \frac{4-3x}{\sqrt{16x^2+1}}$$

   Since $\lim_{x\to\infty}\frac{4-3x}{\sqrt{16x^2+1}} = -\frac{3}{4}$ and $\lim_{x\to-\infty}\frac{4-3x}{\sqrt{16x^2+1}} = \frac{3}{4}$, the horizontal asymptotes are $y = \frac{3}{4}$ and $y = -\frac{3}{4}$. There are no vertical asymptotes since $\frac{4-3x}{\sqrt{16x^2+1}}$ exists for all x values.

2. A man 6 feet tall is walking away from a light pole which is 30 feet high. If the tip of his shadow is moving at a rate equal to the distance between him and the light pole (in feet) then how fast is the man walking when he is 24 feet from the pole?

   Let l be the distance from the man to the lamp pole, x the distance from the lamp pole to the tip of the man's shadow, and t be time. We are told that $\frac{dx}{dt} = l$. We want to know what $\frac{dl}{dt}$ is when $l = 24$. By similar triangles, we obtain

$$\frac{6}{30} = \frac{x-l}{x}$$

   which can be solved to obtain $l = \frac{4}{5}x$. Therefore,

$$\frac{dl}{dt} = \frac{d}{dt}\left(\frac{4}{5}x\right) = \frac{4}{5}\frac{dx}{dt} = \frac{4}{5}l$$

   so when $l = 24$, $\frac{dl}{dt} = \frac{96}{5}$.

3. A spherical snowball is melting at a rate equal to its surface area. How fast is its radius shrinking when its volume is equal to its surface area?

   Let r be the radius of the spherical snowball, V the volume of the sphere, A be the surface area of the snowball, and t be time. We are told that $\frac{dV}{dt} = -A$. We want to know what $-\frac{dr}{dt}$ is when $V = A$. Recall that $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$ so
   $$\frac{dV}{dt} = \frac{d}{dt}(\frac{4}{3}\pi r^3) = 4\pi r^2 \frac{dr}{dt} = A\frac{dr}{dt}.$$
   On the other hand, $\frac{dV}{dt} = -A$, therefore, combining with the previous equation, $\frac{dr}{dt} = -1$ at any moment in time.

4. Two nonnegative numbers are such that the sum of the first number and 3 times the second number equals 10. Find these numbers if the sum of their squares is as small as possible.

   Call the numbers x and y. We know that $x + 3y = 10$ or, in other words, $x = 10 - 3y$. Furthermore, let $f = x^2 + y^2 = (10 - 3y)^2 + y^2$. Notice that the fact that both x and y be nonnegative means that y must lie in the interval $[0, \frac{10}{3}]$. We want to find the absolute minimum of $f(y)$ on the interval $[0, \frac{10}{3}]$.

   Since $f'(y) = -60 + 20y$, the only stationary point of f (i.e. where $f'(y) = 0$) is $y = 3$. There are no singular points of f. Therefore, the only critical point of f is 3. The global minimum of f is then the smallest value of f evaluated on the endpoints (namely $f(0) = 100$, $f(\frac{10}{3}) = \frac{100}{9}$) or $f(3) = 10$. Therefore, the two numbers are $x = 10 - 3(3) = 1$ and $y = 3$.

5. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

   Suppose the length of a side of the square is x and the length of the side of the triangle is y then the area of the square is x^2. The area of the equilateral triangle is $\frac{1}{2}y(\frac{\sqrt{3}}{2}y) = \frac{\sqrt{3}}{4}y^2$. Therefore, their combined area $A = x^2 + \frac{\sqrt{3}}{4}y^2$. Since $4x + 3y = 10$, we have $y = (10 - 4x)/3$. Plugging this into A and simplifying, we get
   $$A(x) = x^2 + \frac{(5 - 2x)^2}{3\sqrt{3}}.$$
   Furthermore, $0 \leq x \leq \frac{5}{2}$ since x must be nonnegative and the largest x can be is if each of its sides is of length $\frac{10}{4} = \frac{5}{2}$.

   In part (a), we want to find the global maximum of $A(x)$ on the interval $[0, \frac{5}{2}]$. We first find the critical numbers of $A(x)$. We observe that
   $$A'(x) = \frac{-4\,(5 - 2\,x)}{3\,\sqrt{3}} + 2\,x.$$
   The stationary points of $A(x)$ come from solving $A'(x) = 0$. Therefore, the only stationary point occurs when
   $$x = \frac{10\,\sqrt{3}}{9 + 4\,\sqrt{3}}.$$

There are no singular points since $A'(x)$ always exists. Therefore, the only critical number is the stationary point.

Now, we calculate

$$A(0) = \frac{25}{3\sqrt{3}} = 4.81125\ldots,$$
$$A(\frac{5}{2}) = \frac{25}{4} = 6.25$$

and

$$A(\frac{10\sqrt{3}}{9+4\sqrt{3}}) = 2.71853\ldots$$

The answer to part (a) is when A is a global maximum on $[0, \frac{5}{2}]$ which occurs when $x = \frac{5}{2}$. In other words, the entire wire is used to make only the square.

The answer to part (b) is when A is global minimum on $[0, \frac{5}{2}]$ which occurs when $x = \frac{10\sqrt{3}}{9+4\sqrt{3}}$. This corresponds to cutting the original wire at position

$$\frac{40\sqrt{3}}{9+4\sqrt{3}}.$$

6. Use Newton's method to find the solution to the equation $\ln(4-x^4) = x$ up to 5 decimal place accuracy.

   Let $f(x) = \ln(4-x^2) - x$ then $f'(x) = \frac{-2x}{4-x^2} - 1$. Therefore, Newton's method gives us the formula

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

   for all $n = 1, 2, 3, \ldots$. We now need only choose x_1 close to a root of f. Notice that $f(1) = -1 + \ln 3 = 0.0986123\ldots$ so let's choose $x_1 = 1$ then $x_2 = 1.059167\ldots$, $x_3 = 1.058007\ldots$, $x_4 = 1.058006\ldots$ Therefore, by rounding up, we obtain the solution 1.05801 up to an accuracy of 5 decimal places.

7. Consider the function

   $$f(x) = 3x^4 - 4x^3 + 20000$$

   (a) On what interval(s) is f increasing? Since

   $$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$$

   $f'(x)$ is positive on the interval $(1, \infty)$.

   (b) On what interval(s) is f concave down? Since

   $$f''(x) = -24x + 36x^2 = 12x(-2+3x)$$

   $f''(x)$ is negative on the interval $(0, \frac{2}{3})$.

   (c) Find the inflection point(s) of f.

   The concavity changes across $x = \frac{2}{3}$ and $x = 0$.

(d) Find the critical points of f.

The stationary points are $x = 0, 1$. There are no singular points.

(e) Find the local maximum (maxima) of f.

There are no local maxima by the first derivative test.

(f) Find the global minimum of f on the interval $[-2, 3]$.

$x = 1$ is a local minimum by the first derivative test. Evaluating f on the endpoints ($f(-2) = 20080$, $f(3) = 20135$) and $f(1) = 19999$, the smallest is 19999 which occurs at $x = 1$.

8. Consider the function

$$f(x) = |x^2 - 5|$$

(a) On what interval(s) is f increasing? We need to calculate $f'(x)$. We recall that

$$|u| = \begin{cases} u, & \text{if } u \geq 0; \\ -u, & \text{if } u < 0 \end{cases}.$$

In our case, $u = x^2 - 5$ and $x^2 - 5 \geq 0$ is equivalent to saying that either $x \geq \sqrt{5}$ or $x \leq -\sqrt{5}$. Similarly, $x^2 - 5 < 0$ is equivalent to $-\sqrt{5} < x < \sqrt{5}$. Therefore, have

$$f(x) = \begin{cases} x^2 - 5, & \text{if } x \geq \sqrt{5} \text{ or } x \leq -\sqrt{5}; \\ -(x^2 - 5), & \text{if } -\sqrt{5} < x < \sqrt{5} \end{cases}.$$

Calculating $f'(x)$, we obtain

$$f'(x) = \begin{cases} 2x, & \text{if } x > \sqrt{5} \text{ or } x < -\sqrt{5}; \\ -2x, & \text{if } -\sqrt{5} < x < \sqrt{5}; \\ \textbf{DNE}, & \text{if } x = \pm\sqrt{5} \end{cases};$$

The stationary points of f occur when $f'(x) = 0$. This is precisely when $x = 0$. The singular points of f are $x = \pm\sqrt{5}$.

Therefore, f is increasing whenever $f'(x) > 0$ which is on the intervals $(-\sqrt{5}, 0)$ or $(\sqrt{5}, \infty)$.

(b) On what interval(s) is f concave down?

$$f''(x) = \begin{cases} 2, & \text{if } x > \sqrt{5} \text{ or } x < -\sqrt{5}; \\ -2, & \text{if } -\sqrt{5} < x < \sqrt{5}; \\ \textbf{DNE}, & \text{if } x = \pm\sqrt{5} \end{cases};$$

f is concave down whenever $f''(x) < 0$ which is on the interval $(-\sqrt{5}, \sqrt{5})$.

(c) Find the inflection point(s) of f.

The inflection points are those points where f changes concavity. This occurs when $x = \pm\sqrt{5}$.

(d) Find the critical points of f.

We did this above: $x = \pm\sqrt{5}, 0$.

(e) Find the local maximum (maxima) of f.

The only local maximum occurs when $x = 0$ and the local maximum is $f(0) = 5$.

(f) Find the global minimum of f on the interval $[-2, 3]$.

$f(-2) = 1$, $f(3) = 4$, $f(0) = 5$, $f(\pm\sqrt{5}) = 0$. Therefore, the global minimum occurs when $x = \pm\sqrt{5}$ and the global minimum is $f(\pm\sqrt{5}) = 0$.

9. Suppose the graph on the following page is of $y = f(x)$

(a) Find the critical numbers of f.

$x = -2, 0, 2$ or $x \geq 6$ are the critical numbers.

(b) On what interval(s) is f increasing?

$(-\infty, -2)$ or $(0, 2)$.

(c) On what interval(s) is f concave down?

$(-2, 0)$ or $(0, 4)$.

(d) Find the values of x on the interval $(-\infty, \infty)$ where f has a local minimum.

$x = 0$ or $x \geq 6$.

(e) Find the values of x on the interval $[0, 4]$ where f has a global minimum.

$x = 0$ or $x = 4$.

(f) Find the x values of all inflection points of f.

$x = 4$.

10. Suppose the graph on the following page is of $y = f'(x)$ (**NOT** $f(x)$).

(a) Find the critical numbers of f.

The stationary points are where $f'(x) = 0$ which occurs when $x = -4, 0, 4$. The singular points are where $f'(x)$ does not exist. There are no such points in this case. Therefore, $x = -4, 0, 4$ are the critical numbers.

(b) On what interval(s) is f increasing?

$f(x)$ is increasing on the intervals $(-4, 0)$ and $(0, 4)$.

(c) On what interval(s) is f concave down?

$f(x)$ is concave down on $(-2, 0)$ and $(2, 6)$.

(d) Find the values of x on the interval $(-\infty, \infty)$ where f has a local minimum.

$f(x)$ has a local minimum when $x = -4$ only.

(e) Find the values of x on the interval $[0, 4]$ where f has a global minimum.

$x = 0$ is the global minimum since $f(x)$ is increasing on $(0, 4)$.

(f) Find the x values of all inflection points of f.

$x = -2, 0, 2$