Math 563A1, Take Home Midterm Exam, October 20, 2003
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This exam is due at 5pm at my office MCS 234 on Monday October 27. (Please slide it under my door if I'm not around.) The exam is open book and you may work with others although you must write up the solutions yourself. Good luck!

1. (10 points) Let α be a unit-speed curve which lies on a sphere of radius R with center located at p in $\mathbf{R}^3$. Show that if $\tau \neq 0$ then

$$\alpha(s) - p = -\frac{1}{\kappa}N - \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}B$$

   and

$$R^2 = \left(\frac{1}{\kappa}\right)^2 + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)^2.$$

   On the other hand, show that if $\left(\frac{1}{\kappa}\right)' \neq 0$ and $\left(\frac{1}{\kappa}\right)^2 + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)^2$ is constant then a (unit speed) curve α lies on a sphere.

2. (10 points) Let $\alpha(t)$ be a unit speed curve in $\mathbf{R}^3$ for all t with positive torsion. Let τ_α, κ_α, T_α, N_α, B_α denote the torsion, curvature, unit tangent, normal, and binormal vectors of α.

   Consider the curve $\beta(s) := \int_0^s B_\alpha(t)dt$. Show that $\kappa_\beta = \tau_\alpha$, $\tau_\beta = \kappa_\alpha$, $T_\beta = B_\alpha$, $N_\beta = -N_\alpha$, and $B_\beta = T_\alpha$.

   Now show that if α is a circular helix then so is β.

3. (10 points) Let $\alpha : I \to \mathbf{R}$ be a regular curve parametrized by arc length whose curvature is nowhere vanishing. Let

$$x(s,v) = \alpha(s) + r\,(n(s)\cos(v) + b(s)\sin(v))$$

   (where r is a nonzero constant and s belongs to I) be a parametrized surface (the *tube* of radius r around α), where n is the normal vector and b is the binormal vector of α. Show that if x is regular then its unit normal vector is

$$N(s,v) = -(n(s)\cos(v) + b(s)\sin(v)).$$

4. (30 points) Define a surface S as the image of the map

$$\sigma(u,v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$$

   for all (u,v) such that $u^2 + v^2 < 3$.

(a) Prove that σ is a regular, single surface patch.

Hint: Use polar coordinates $u = r\cos\theta$ and $v = r\sin\theta$ and show that

$$x^2 + y^2 + \frac{4}{3}z^2 = \frac{1}{9}r^2(3 + r^2)^2$$

holds where (x, y, z) belongs to S. Then show that the equality implies that points in the (u, v)-plane on different circles about $(0, 0)$ cannot be mapped to the same point.

(b) Calculate the first fundamental form of σ.

(c) Let D denote subset of S consisting of those points $\sigma(u, v)$ such that (u, v) belongs to the set $[-1, 1] \times [-1, 1]$. Find the area of D.

(d) Choose an orientation for S. Consider the map $f : S \to S'$, where S' is the unit sphere about the origin in $\mathbf{R}^3$, defined by

$$f(\sigma(u, v)) := N(u, v)$$

where $N(u, v)$ denotes the unit normal to S at the point $\sigma(u, v)$. Show that f is a conformal map.