MA 129 Discussion Section 11/8/2004

Fourier Series

```
a[n_] := a[n] = 1/Pi Integrate[f[x] Cos[n x],{x,-Pi,Pi}]
```

```
b[n_] := b[n] = 1/Pi Integrate[f[x] Sin[n x],{x,-Pi,Pi}]
```

```
s[x_,m_] := a[0]/2 + Sum[a[n] Cos[n x] + b[n] Sin[n x],{n,1,m}]
```

```
f[x_] := x^2
```

■ Here's the partial sum up to n=1

```
s[x,1]
```

$$\frac{\pi^2}{3} - 4\,\mathrm{Cos}[x]$$

■ Here is the partial sum up to n=5

```
s[x,5]
```

$$\frac{\pi^2}{3} - 4\,\mathrm{Cos}[x] + \mathrm{Cos}[2\,x] - \frac{4}{9}\,\mathrm{Cos}[3\,x] + \frac{1}{4}\,\mathrm{Cos}[4\,x] - \frac{4}{25}\,\mathrm{Cos}[5\,x]$$

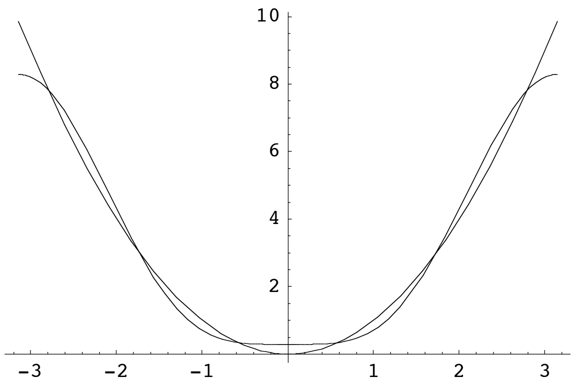
■ Here is the partial sum up to n=100

```
s[x,100]
```

$$\frac{\pi^2}{3} - 4\,\text{Cos}[x] + \text{Cos}[2\,x] - \frac{4}{9}\,\text{Cos}[3\,x] + \frac{1}{4}\,\text{Cos}[4\,x] - \frac{4}{25}\,\text{Cos}[5\,x] + \frac{1}{9}\,\text{Cos}[6\,x] - \frac{4}{49}\,\text{Cos}[7\,x] + \frac{1}{16}\,\text{Cos}[8\,x] - \frac{4}{81}\,\text{Cos}[9\,x] + \frac{1}{25}\,\text{Cos}[10\,x] - \frac{4}{121}\,\text{Cos}[11\,x] + \frac{1}{36}\,\text{Cos}[12\,x] - \frac{4}{169}\,\text{Cos}[13\,x] + \frac{1}{49}\,\text{Cos}[14\,x] - \frac{4}{225}\,\text{Cos}[15\,x] + \frac{1}{64}\,\text{Cos}[16\,x] - \frac{4}{289}\,\text{Cos}[17\,x] + \frac{1}{81}\,\text{Cos}[18\,x] - \frac{4}{361}\,\text{Cos}[19\,x] + \frac{1}{100}\,\text{Cos}[20\,x] - \frac{4}{441}\,\text{Cos}[21\,x] + \frac{1}{121}\,\text{Cos}[22\,x] - \frac{4}{529}\,\text{Cos}[23\,x] + \frac{1}{144}\,\text{Cos}[24\,x] - \frac{4}{625}\,\text{Cos}[25\,x] + \frac{1}{169}\,\text{Cos}[26\,x] - \frac{4}{729}\,\text{Cos}[27\,x] + \frac{1}{196}\,\text{Cos}[28\,x] - \frac{4}{841}\,\text{Cos}[29\,x] + \frac{1}{225}\,\text{Cos}[30\,x] - \frac{4}{961}\,\text{Cos}[31\,x] + \frac{1}{256}\,\text{Cos}[32\,x] - \frac{4\,\text{Cos}[33\,x]}{1089} + \frac{1}{289}\,\text{Cos}[34\,x] - \frac{4\,\text{Cos}[35\,x]}{1225} + \frac{1}{324}\,\text{Cos}[36\,x] - \frac{4\,\text{Cos}[37\,x]}{1369} + \frac{1}{361}\,\text{Cos}[38\,x] - \frac{4\,\text{Cos}[39\,x]}{1521} + \frac{1}{400}\,\text{Cos}[40\,x] - \frac{4\,\text{Cos}[41\,x]}{1681} + \frac{1}{441}\,\text{Cos}[42\,x] - \frac{4\,\text{Cos}[43\,x]}{1849} + \frac{1}{484}\,\text{Cos}[44\,x] - \frac{4\,\text{Cos}[45\,x]}{2025} + \frac{1}{529}\,\text{Cos}[46\,x] - \frac{4\,\text{Cos}[47\,x]}{2209} + \frac{1}{576}\,\text{Cos}[48\,x] - \frac{4\,\text{Cos}[49\,x]}{2401} + \frac{1}{625}\,\text{Cos}[50\,x] - \frac{4\,\text{Cos}[51\,x]}{2601} + \frac{1}{676}\,\text{Cos}[52\,x] - \frac{4\,\text{Cos}[53\,x]}{2809} + \frac{1}{729}\,\text{Cos}[54\,x] - \frac{4\,\text{Cos}[55\,x]}{3025} + \frac{1}{784}\,\text{Cos}[56\,x] - \frac{4\,\text{Cos}[57\,x]}{3249} + \frac{1}{841}\,\text{Cos}[58\,x] - \frac{4\,\text{Cos}[59\,x]}{3481} + \frac{1}{900}\,\text{Cos}[60\,x] - \frac{4\,\text{Cos}[61\,x]}{3721} + \frac{1}{961}\,\text{Cos}[62\,x] - \frac{4\,\text{Cos}[63\,x]}{3969} + \frac{\text{Cos}[64\,x]}{1024} - \frac{4\,\text{Cos}[65\,x]}{4225} + \frac{\text{Cos}[66\,x]}{1089} - \frac{4\,\text{Cos}[67\,x]}{4489} + \frac{\text{Cos}[68\,x]}{1156} - \frac{4\,\text{Cos}[69\,x]}{4761} + \frac{\text{Cos}[70\,x]}{1225} - \frac{4\,\text{Cos}[71\,x]}{5041} + \frac{\text{Cos}[72\,x]}{1296} - \frac{4\,\text{Cos}[73\,x]}{5329} + \frac{\text{Cos}[74\,x]}{1369} - \frac{4\,\text{Cos}[75\,x]}{5625} + \frac{\text{Cos}[76\,x]}{1444} - \frac{4\,\text{Cos}[77\,x]}{5929} + \frac{\text{Cos}[78\,x]}{1521} - \frac{4\,\text{Cos}[79\,x]}{6241} + \frac{\text{Cos}[80\,x]}{1600} - \frac{4\,\text{Cos}[81\,x]}{6561} + \frac{\text{Cos}[82\,x]}{1681} - \frac{4\,\text{Cos}[83\,x]}{6889} + \frac{\text{Cos}[84\,x]}{1764} - \frac{4\,\text{Cos}[85\,x]}{7225} + \frac{\text{Cos}[86\,x]}{1849} - \frac{4\,\text{Cos}[87\,x]}{7569} + \frac{\text{Cos}[88\,x]}{1936} - \frac{4\,\text{Cos}[89\,x]}{7921} + \frac{\text{Cos}[90\,x]}{2025} - \frac{4\,\text{Cos}[91\,x]}{8281} + \frac{\text{Cos}[92\,x]}{2116} - \frac{4\,\text{Cos}[93\,x]}{8649} + \frac{\text{Cos}[94\,x]}{2209} - \frac{4\,\text{Cos}[95\,x]}{9025} + \frac{\text{Cos}[96\,x]}{2304} - \frac{4\,\text{Cos}[97\,x]}{9409} + \frac{\text{Cos}[98\,x]}{2401} - \frac{4\,\text{Cos}[99\,x]}{9801} + \frac{\text{Cos}[100\,x]}{2500}$$

■ Let's plot x^2 against the partial sum s[x,2]

```
Plot[{x^2,s[x,2]},{x,-Pi,Pi}]
```

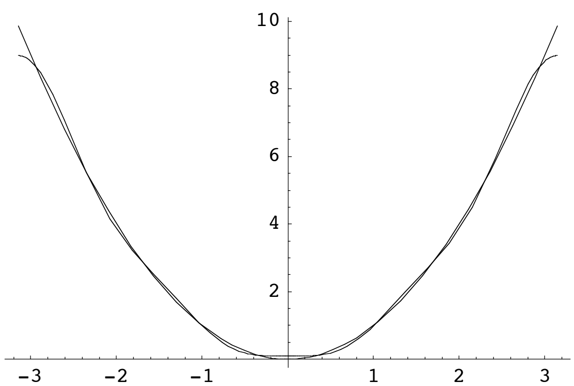


- Graphics -

```
s[x,2]
```

$$\frac{\pi^2}{3} - 4\,\mathrm{Cos}[x] + \mathrm{Cos}[2\,x]$$

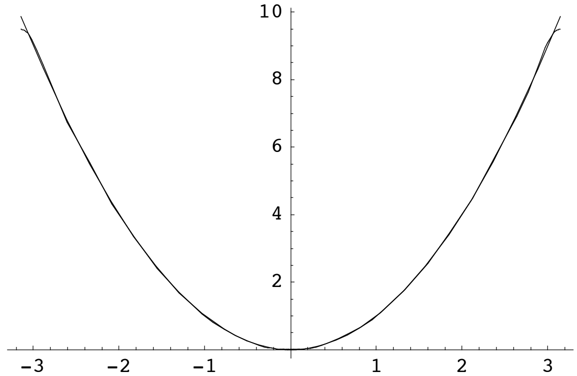
```
Plot[{x^2,s[x,4]},{x,-Pi,Pi}]
```



- Graphics -

Here's a plot of x^2 against the partial sum out to n=10. They're VERY close!

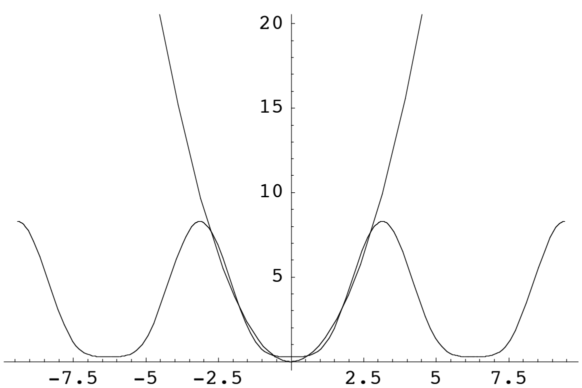
```
Plot[{x^2,s[x,10]},{x,-Pi,Pi}]
```



- Graphics -

■ **But wait a minute, x^2 and the partial sum s[x,2] are not close when |x|>Pi. The series is naturally periodic with period 2 Pi since both Sin[n x] and Cos[n x] are the same if we replace x by x + 2 Pi where n is an integer.**

```
Plot[{x^2,s[x,2]},{x,-3 Pi,3 Pi}]
```



- Graphics -

■ **Now, let's restart the kernel so as to forget all previous settings.**

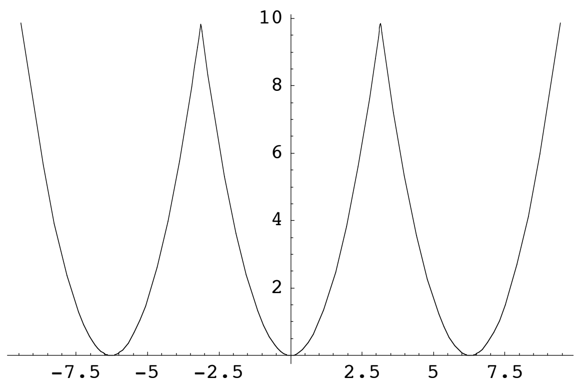
```
In[1]:= a[n_] := a[n] = 1/Pi Integrate[f[x] Cos[n x],{x,-Pi,Pi}]

In[2]:= b[n_] := b[n] = 1/Pi Integrate[f[x] Sin[n x],{x,-Pi,Pi}]

In[3]:= s[x_,m_] := a[0]/2 + Sum[a[n] Cos[n x] + b[n] Sin[n x],{n,1,m}]

In[4]:= f[x_] := x^2
```

Now, g[x] is the same as x^2 when |x| <= Pi and then is extended to a period function with period 2 Pi outside that region. One can write that as a formula as follows:

```
In[5]:= g[x_] := (-Pi + Mod[x+Pi,2 Pi])^2
```

Here's a plot of g[x] where |x| <= 3 Pi.

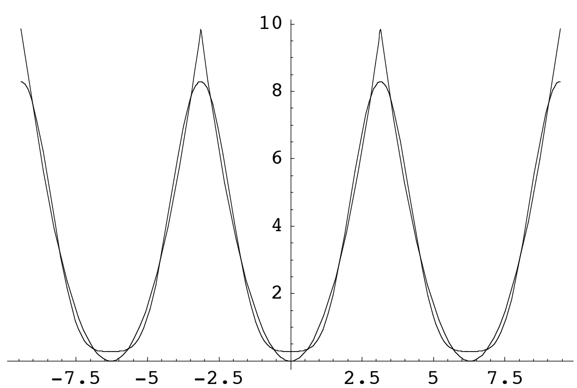
```
In[6]:= Plot[g[x],{x,-3 Pi , 3 Pi}]
```




```
Out[6]= - Graphics -
```

■ Now, let's plot g[s] against the partial sum s[x,2]. It's not a bad fit.

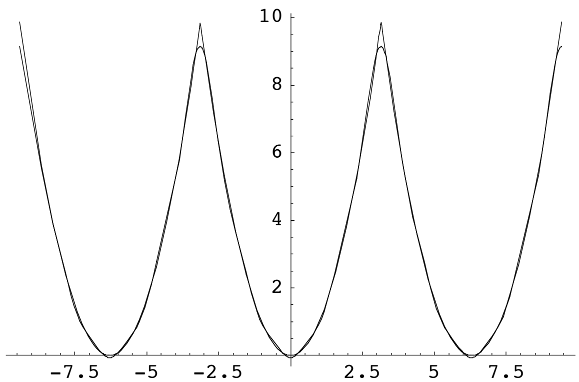
```
In[7]:= Plot[{g[x], s[x, 2]}, {x, -3 Pi, 3 Pi}]
```




```
Out[7]= - Graphics -
```

■ Let's plot g[x] against s[x,5] where |x| <= 3 Pi. This is clearly a better fit.

```
In[8]:= Plot[{g[x],s[x,5]},{x,-3 Pi, 3 Pi}]
```



Out[8]= - Graphics -

■ Here's the Fourier series out to n=10.

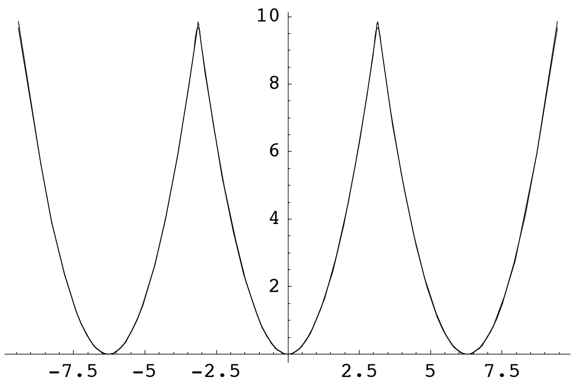
```
In[10]:= s[x,20]
```

Out[10]=

$$\frac{\pi^2}{3} - 4\,\mathrm{Cos}[x] + \mathrm{Cos}[2\,x] - \frac{4}{9}\,\mathrm{Cos}[3\,x] + \frac{1}{4}\,\mathrm{Cos}[4\,x] - \frac{4}{25}\,\mathrm{Cos}[5\,x] + \frac{1}{9}\,\mathrm{Cos}[6\,x] - \frac{4}{49}\,\mathrm{Cos}[7\,x] + \frac{1}{16}\,\mathrm{Cos}[8\,x] - \frac{4}{81}\,\mathrm{Cos}[9\,x] + \frac{1}{25}\,\mathrm{Cos}[10\,x] - \frac{4}{121}\,\mathrm{Cos}[11\,x] + \frac{1}{36}\,\mathrm{Cos}[12\,x] - \frac{4}{169}\,\mathrm{Cos}[13\,x] + \frac{1}{49}\,\mathrm{Cos}[14\,x] - \frac{4}{225}\,\mathrm{Cos}[15\,x] + \frac{1}{64}\,\mathrm{Cos}[16\,x] - \frac{4}{289}\,\mathrm{Cos}[17\,x] + \frac{1}{81}\,\mathrm{Cos}[18\,x] - \frac{4}{361}\,\mathrm{Cos}[19\,x] + \frac{1}{100}\,\mathrm{Cos}[20\,x]$$

■ Plotting g[x] against s[x,20], we see that they are very, very close.

```
In[9]:= Plot[{g[x],s[x,20]},{x,-3 Pi, 3 Pi}]
```




```
Out[9]= - Graphics -
```