Math 721A1, Final Exam
Differential Topology I

(1) Suppose that M is a compact, orientable n-manifold (without boundary) and let θ be an $(n-1)$-form on M. Show that $d\theta$ must vanish at some point.

(2) Let M be a smooth n-manifold and α a closed k-form on M where $k \geq 1$. For all p, let $\Delta_p := \{X \in T_pM \mid \iota(X)\alpha = 0\}$ where ι is the interior product. Furthermore, suppose that the dimension of Δ_p is r (a constant) for all p.
   (a) Prove that $p \mapsto \Delta_p$, denoted by Δ, is a smooth, rank r integrable distribution.
   (b) Let $M = \mathbb{R}^n - \{0\}$ and $\alpha = \sum_{i=1}^n x^i dx^i \in \Omega^1(M)$. Find the leaves of the foliation associated to Δ. Why did we omit the point $\{0\}$ in the definition of M?

(3) Let M be a smooth manifold. Let $\pi_i : M \times M \to M$ be the projection onto the i-th factor for $i = 1, 2$. Let $\Delta : M \to M \times M$ be the diagonal map $\Delta(m) = (m, m)$. Prove that for $[\omega]$ in $\Omega^k(M)$ and $[\psi]$ in $\Omega^l(M)$,

$$[\omega \wedge \psi] = \Delta^*(\pi_1^*[\omega] \wedge \pi_2^*[\psi]).$$

(4) Let n be an integer and $R > 0$, let $c_{R,n} : [0,1] \to \mathbb{R}^2 - \{\,0\,\}$ be defined by

$$c_{R,n}(t) = (R\cos(2n\pi t), R\sin(2n\pi t)).$$

   (a) Show that there exists a singular 2-cube $c : [0,1]^2 \to \mathbb{R} - \{0\}$ such that $c_{R_1,n} - c_{R_2,n} = \partial c$.
   (b) If $c : [0,1] \to \mathbb{R}^2 - \{0\}$ is any curve with $c(0) = c(1)$, show that there is some n such that $c - c_{1,n}$ is a boundary in $\mathbb{R}^2 - \{0\}$.
   (c) Show that n is unique. n is called the *winding number* of c around 0.

(5) Let M be a smooth $2n$-manifold and $\omega \in \Omega^2(M)$ such that
   Closed: $d\omega = 0$.
   Nondegenerate: For all p in M and X, Y in T_pM, if $\omega(X, Y) = 0$ for all X then $Y = 0$.

   (M, ω) is called a *symplectic manifold.*
   (a) Prove that (M, ω) has a canonical orientation.
   (b) For all f in $C^\infty(M)$, let X_f denote the vector field on M defined by the equation

$$\omega^\flat(X_f) + df = 0$$

   where $\omega^\flat : TM \to T^*M$ via $\omega^\flat(Y) := \omega(Y, \cdot)$. ($X_f$ is called the *Hamiltonian vector field associated to* f.) Prove that $L_{X_f}\omega = 0$.
   (c) Let X be a vector field on M such that $L_X\omega = 0$. (Such an X is called a *symplectic vector field.*) Prove that there exists an f in $C^\infty(M)$ such that $X = X_f$ iff that a certain cohomology class on M vanishes.
   (d) The *Poisson bracket* is the map $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ defined by $(f, g) \mapsto \{f, g\}$ where

$$\{f, g\} := \omega(X_f, X_g).$$

   Prove that $\{f, g\} = -\{g, f\}$ and $\{fg, h\} = f\{g, h\} + g\{f, h\}$ for all f, g, h in $C^\infty(M)$.
   (e) Prove that for all f, g, h in $C^\infty(M)$,

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$