Math 721A1, Homework #1
Differential Topology I

(1) Consider the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$. Show that f is not a diffeomorphism although it is a smooth, bijection.

(2) Prove that a function $f : M \to N$ is C^∞ if and only if $g \circ f$ is C^∞ for every C^∞ function $g : N \to \mathbb{R}$.

(3) Let $\mathrm{RP}^2 = \{[x, y, z] \,|, \ (x, y, z) \in S^2\}$ where we have the equivalence relation

$$[x, y, z] = [-x, -y, -z].$$

Let $g : \mathrm{RP}^2 \to \mathbb{R}^3$ be the map

$$g([x, y, z]) := (yz, xz, xy).$$

Show that g fails to be an immersion at 6 points.

(4) Prove the following proposition: Let M and N be smooth m and n dimensional manifolds, resp. If $f : M \to N$ is a smooth, rank k map on a neighborhood of $f^{-1}(y)$ then $f^{-1}(y)$ is a closed submanifold of M of dimension $m-k$ or is empty. In particular, if y is a regular value of f then $f^{-1}(y)$ is an (m-n)-dimensional submanifold of M (or is empty).

(5) (a) The set of all non-singular $n\times n$ matrices with real entries is called $\mathrm{GL}(n, \mathbb{R})$, the *general linear group*. It is a smooth n^2 dimensional manifold as it is an open subset of $\mathbb{R}^{n^2}$. Prove that the matrix multiplication map

$$\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$$

taking $(A, B) \mapsto AB$ is a smooth map. A smooth manifold G which is also a group where the group multiplication $G \times G \to G$ is a smooth map is called a *Lie group*.

(b) Let $\mathrm{SL}(n, \mathbb{R})$, the *special linear group*, be the subset of $\mathrm{GL}(n, \mathbb{R})$ consisting of those elements which have determinant 1. Prove that $\mathrm{SL}(n, \mathbb{R})$ is a closed $(n^2 - 1)$ dimensional submanifold of $\mathrm{GL}(n, \mathbb{R})$. Furthermore, prove that it is a Lie group. We say that $\mathrm{SL}(n, \mathbb{R})$ is a *(closed) Lie subgroup of* $\mathrm{GL}(n, \mathbb{R})$.

(6) Show that a smooth map $f : M \to N$ is an immersion if and only if f_* is an injective map $T_pM \to T_{f(p)}N$ for all p in M.

(7) Consider S^2, the unit sphere about the origin in $\mathbb{R}^3$. Consider the curve $c : \mathbb{R} \to S^2$ defined by

$$c(t) := (\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}).$$

Let $x_{3,+}$ be a coordinate chart on S^2 defined (as in class) by

$$x_{3,+}(u, v, w) := (u, v).$$

and let $x_{1,+}$ be a coordinate chart on S^2 defined (as in class) by

$$x_{1,+}(u, v, w) := (v, w).$$

(a) Write the tangent vector Y to the curve $c(t)$ at $t = 0$ in $x_{3,+}$ coordinates.
(b) Write Y in $x_{1,+}$ coordinates.

(8) Show that a rank k vector bundle $\pi : E \to B$ is a trivial bundle if and only if it has k sections $s_1, \dots, s_k$ such that $\{ s_1(p), \dots, s_k(p)\}$ are linearly independent for all p in B.

(9) Show that for any smooth manifold M, TM is an orientable smooth manifold.