

Math 721A1, Homework #3
Differential Topology I

- (1) Show that n functions $f_1, \dots, f_n : M \rightarrow \mathbb{R}$ form a coordinate system in a neighborhood of p in M if and only if

$$(df_1 \wedge df_2 \cdots \wedge df_n)_p \neq 0.$$

- (2) Let v_1, \dots, v_n be a basis for V and v_1^*, \dots, v_n^* be its dual basis, and w_1, \dots, w_n be another basis for V with its dual basis w_1^*, \dots, w_n^* such that

$$w_i = \sum_{j=1}^n \alpha_{ji} v_j$$

for all $i = 1, \dots, n$. Show that

$$\det(\alpha_{ij}) w_1^* \wedge \cdots \wedge w_n^* = v_1^* \wedge \cdots \wedge v_n^*.$$

- (3) If ω is a $(k+1)$ -form ($k \geq 0$) then $\iota_X \omega$ is the k -form defined by

$$(\iota_X \omega)(X_1, \dots, X_k) := \omega(X, X_1, \dots, X_k).$$

$\iota_X \omega$ is called the interior product (or contraction) of ω with X . (Define $\iota_X f := 0$ for any smooth function f .) Show the following.

- (a) Show that for any X , k -form ω_1 , and l -form ω_2 ,

$$\iota_X(\omega_1 \wedge \omega_2) = (\iota_X \omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (\iota_X \omega_2)$$

- (b) Show that

$$L_X(\omega_1 \wedge \omega_2) = (L_X \omega_1) \wedge \omega_2 + \omega_1 \wedge (L_X \omega_2)$$

- (c) Show that

$$L_X \omega = d(\iota_X \omega) + \iota_X(d\omega).$$

- (d) Using the previous, show that

$$d(L_X \omega) = L_X(d\omega).$$

- (4) Consider \mathbb{R}^3 with the standard Euclidean metric so that $T_p \mathbb{R}^3$ is identified with $T_p^* \mathbb{R}^3$. Let $X = \sum_{i=1}^3 a^i \frac{\partial}{\partial x^i}$ be a vector field on \mathbb{R}^3 and $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ a function. Define the gradient of f by

$$\text{grad } f := \sum_{i=1}^n \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i},$$

the divergence of X by

$$\text{div } X := \sum_{i=1}^n \frac{\partial a^i}{\partial x^i},$$

and the curl of X by

$$\text{curl } X := \left(\frac{\partial a^3}{\partial x^2} - \frac{\partial a^2}{\partial x^3} \right) \frac{\partial}{\partial x^1} + \left(\frac{\partial a^1}{\partial x^3} - \frac{\partial a^3}{\partial x^1} \right) \frac{\partial}{\partial x^2} + \left(\frac{\partial a^2}{\partial x^1} - \frac{\partial a^1}{\partial x^2} \right) \frac{\partial}{\partial x^3}.$$

Define the differential forms on \mathbb{R}^3 ,

$$\omega_X = a^1 dx^1 + a^2 dx^2 + a^3 dx^3$$

and

$$\eta_X = a^1 dx^2 \wedge dx^3 + a^2 dx^3 \wedge dx^1 + a^3 dx^1 \wedge dx^2.$$

(a) Show that

$$\begin{aligned}df &= \omega_{\text{grad}f}, \\d(\omega_X) &= \eta_{\text{curl}X} \\d(\eta_X) &= (\text{div}X) dx^1 \wedge dx^2 \wedge dx^3.\end{aligned}$$

(b) Conclude that

$$\text{curl}(\text{grad}f) = 0$$

and

$$\text{div}(\text{curl}X) = 0.$$

- (c) In the movie, *A Beautiful Mind*, John Nash, played by Russell Crowe, assigns a problem to his calculus class at MIT which he writes out on the board. State the problem in modern differential geometric language.
- (5) Prove that any smooth manifold M has a Riemannian metric. *Hint*: Use the existence of partitions of unity and the existence of the standard dot product on open subsets of \mathbb{R}^n to construct a Riemannian metric on M .