

Math 123, Practice Exam Solutions for Exam #1, October 3, 1999

1. Calculate the following:

(a)

$$\lim_{x \rightarrow -1} \frac{x^2 - 4x - 5}{x + 1} = \lim_{x \rightarrow -1} \frac{(-5 + x)(1 + x)}{x + 1} = \lim_{x \rightarrow -1} (-5 + x) = -6$$

(b)

$$\lim_{x \rightarrow 3} \frac{5x^2}{2x - 1} = \frac{5(3)^2}{2(3) - 1} = 9$$

(c)

$$\lim_{x \rightarrow 2^+} \frac{4 - x^2}{|2 - x|} = \lim_{x \rightarrow 2^+} \frac{4 - x^2}{-(2 - x)} = \lim_{x \rightarrow 2^+} \frac{(2 - x)(x + 2)}{-(2 - x)} = \lim_{x \rightarrow 2^+} -(x + 2) = -4$$

(d)

$$\lim_{x \rightarrow 1} \sqrt{\frac{2x^3 - 3x + 5}{2 - x}} = \sqrt{\frac{2 \cdot 1^3 - 3 \cdot 1 + 5}{2 - 1}} = \sqrt{4} = 2$$

(e)

$$\lim_{x \rightarrow 3} f(x) \text{ where } f(x) = \begin{cases} x^2 & \text{if } x > 3 \\ 8 & \text{if } x = 3 \\ 12 - x & \text{if } x < 3 \end{cases}$$

Notice that  $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (12 - x) = 9$  and  $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} x^2 = 9$ . Since both one-sided limits agree with the value 9,  $\lim_{x \rightarrow 3} f(x) = 9$ .

(f)

$$\lim_{x \rightarrow 3^-} \frac{x + 3}{x^2 - 9} = \lim_{x \rightarrow 3^-} \frac{x + 3}{(x - 3)(x + 3)} = \lim_{x \rightarrow 3^-} \frac{1}{x - 3} = -\infty$$

2. Consider the function

$$f(x) = \begin{cases} x - c, & \text{if } x > 2; \\ 3x^2, & \text{if } x \leq 2 \end{cases}$$

where  $c$  is a real number.

- (a) What value of  $c$  makes the function  $f$  continuous everywhere? The only place where the function  $f$  might not be continuous is at  $x = 2$  but we can avoid this if we choose  $c$  so that the graphs of  $y = x - c$  and  $y = 3x^2$  agree at  $x = 2$ , i.e. so there is no jump across  $x = 2$ . Solving  $2 - c = 3 \cdot 2^2$  means that  $c = -10$ .
- (b) If  $x > 2$  then  $f(x) = x - c \Rightarrow f'(x) = 1 \Rightarrow f'(7) = 1$
- (c) If  $x < 2$  then  $f(x) = 3x^2 \Rightarrow f'(x) = 6x \Rightarrow f'(-1) = -6$
- (d) Call  $f_1(x) = x - c$  and  $f_2(x) = 3x^2$ . Notice that  $f'(2)$  does not exist (even when  $c = -10$ ) since  $f_1'(2) = 1$  which does not agree with  $f_2'(2) = 6 \cdot 2 = 12$ .

3. Compute the derivative of the following functions:

(a)

$$f(x) = \pi^4 \Rightarrow f'(x) = 0$$

(b)

$$f(x) = 3x^5 - x^2 + 9 \Rightarrow f'(x) = 15x^4 - 2x$$

(c)

$$f(x) = \frac{2}{x^2} - 3\sqrt{x} = 2x^{-2} - 3x^{\frac{1}{2}} \Rightarrow f'(x) = -4x^{-3} - \frac{3}{2}x^{-\frac{1}{2}}$$

(d)

$$f(x) = \frac{x^2}{2x-3} \Rightarrow f'(x) = \frac{(2x-3)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(2x-3)}{(2x-3)^2} = \frac{(2x-3)(2x) - x^2(2)}{(2x-3)^2} = \frac{2x^2 - 6x}{(2x-3)^2}$$

(e)

$$f(x) = x^2e^x \Rightarrow f'(x) = e^x\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(e^x) = 2xe^x + x^2e^x$$

(f)

$$f(t) = t \sin t \Rightarrow f'(t) = t\frac{d}{dt}(\sin t) + \sin t\frac{d}{dt}(t) = t \cos t + \sin t$$

4. Find the equation for the tangent line to the curve  $y = f(x)$  where through the point  $(1, -3)$  where

$$f(x) = x^8 - 4x.$$

$f'(x) = 8x^7 - 4$ . The slope of the tangent line is  $f'(1) = 4$ . Therefore the equation for this tangent line is

$$\frac{y - (-3)}{x - 1} = 4$$

or, solving for  $y$ ,  $y = 4x - 7$ .

5. Suppose an object is moving along the real line with its position at time  $t$  given by the function  $s(t) = \frac{1}{3}t^3 - 3t^2 - 7t + 10$ .

(a) The object is at rest when the velocity  $v(t) = 0$  but  $v(t) = s'(t) = t^2 - 6t - 7 = (t-7)(t+1) = 0$ . The solution is  $t = -1$  or  $t = 7$ .

(b) The object is decelerating when the acceleration  $a(t) < 0$  but  $a(t) = v'(t) = 2t - 6 < 0$  when  $t < 3$ .

(c)  $v(2) = -15$ .

6. Consider the function

$$f(x) = x^2 - \sin x.$$

Using your calculator, find the approximate value(s) of  $x$  (out to two decimal places) at which the tangent line to the graph  $y = f(x)$  is horizontal.

We are solving for the equation  $0 = f'(x) = 2x - \cos x$ . Using the calculator, we can see that an approximate solution is  $x \approx .45$ .

7. Consider the graph of  $y = f(x)$  on the next page (figure 1).

(a) Where is  $f$  undefined?

$$x = -1, 0, 3$$

(b) Where is  $f$  not continuous?

$$x = -5, -1, 0, 3$$

(c) Where is  $f$  not differentiable?

$$x = -5, -3, -2, -1, 0, 3, 5$$

(d) On what interval(s) is  $f'$  positive? Where does  $f'$  vanish?  $f'$  is positive on the intervals  $(-5, -4)$ ,  $(-2, -1)$ ,  $(2, 3)$ ,  $(3, 5)$ .  $f'$  vanishes at  $x = -4, 2$

(e) On what interval(s) is  $f$  concave down?  $(-\infty, -5), (-5, -3), (-1, 0), (3, 5)$ .

(f) What are

i.

$$f'(6) = \frac{0 - 3}{7.25 - 5} = -\frac{3}{2.25}$$

ii.

$$\lim_{x \rightarrow -5^+} f(x) = -2$$

iii.

$$\lim_{x \rightarrow -1} f(x) = -1$$

iv.

$$\lim_{x \rightarrow -\infty} f(x) = 0$$