Math 123, Practice Exam #3 Solutions, December 8, 1999

1. Find the following:

(a)

$$\int (5t^3 - \frac{7}{t^4} + 6\sqrt{t} - 4\sin t)\,dt = \int (5t^3 - 7t^{-4} + 6t^{\frac{1}{2}} - 4\sin t)\,dt$$
$$= \frac{5}{4}t^4 + \frac{7}{3}t^{-3} + 4t^{\frac{3}{2}} + 4\cos t + C$$

(b) Let $u = \sin x$ then $du = \frac{du}{dx}dx = \cos x dx$ and

$$\int \sin^{100}(x)\cos(x)\,dx = \int u^{100}\,du = \frac{u^{101}}{101} + C = \frac{\sin^{101}(x)}{101} + C$$

(c) Let $u = x^3 - 8x^2 + 5x + 3$ then $du = \frac{du}{dx}dx = (3x^2 - 16x + 5)\,dx$ then

$$\int \frac{3x^2 - 16x + 5}{\sqrt{x^3 - 8x^2 + 5x + 3}}\,dx = \int (3x^2 - 16x + 5)(x^3 - 8x^2 + 5x + 3)^{-\frac{1}{2}}\,dx$$
$$= \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} + C = 2(x^3 - 8x^2 + 5x + 3)^{\frac{1}{2}} + C$$

(d) Let $u = x^2 - 3x$ then $\frac{du}{dx} = (2x - 3)$ or $\frac{1}{2}\frac{du}{dx} = (x - \frac{3}{2})\,dx$ and

$$\int (x - \frac{3}{2})\sin(x^2 - 3x)\,dx = \int \sin(u)\,\frac{1}{2}\,du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2 - 3x) + C$$

(e)

$$\int x\ln x^3\,dx = \int x(3\ln x)\,dx = 3\int x\ln x\,dx = 3\int u\,dv$$

where $u = \ln x$ and $v = \frac{x^2}{2}$. Using integration by parts, we obtain

$$\int x\ln x^3\,dx = 3\left(\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{d}{dx}(\ln x)\,dx\right) = \frac{3x^2}{2}\ln x - \frac{3x^2}{4} + C$$

(f)

$$\int_1^2 \frac{1 - x^3}{x^2}\,dx = \int_1^2 \left(\frac{1}{x^2} - x\right)\,dx = \left(-x^{-1} - \frac{x^2}{2}\right)\Big|_1^2 = -1$$

(g) Choose $u = \tan x$ then $du = \sec^2 x\,dx$. Also, when $x = 0$ then $u = \tan 0 = 0$ and when $x = \frac{\pi}{4}$ then $u = \tan\frac{\pi}{4} = 1$ so

$$\int_0^{\frac{\pi}{4}} \tan^5 x\sec^2 x\,dx = \int_0^1 u^5\,dx = \frac{u^6}{6}\Big|_0^1 = \frac{1}{6}$$

(h)

$$\frac{d}{dx}\int_3^x \sqrt{t^3 + 1}\,dt = \sqrt{x^3 + 1}$$

by using the fundamental theorem of calculus.

(i)

$$\frac{d}{dx}\int_{2x}^{x^3} \sin t^2\,dt = \frac{d}{dx}\left(\int_1^{x^3} \sin t^2\,dt + \int_{2x}^1 \sin t^2\,dt\right) = \frac{d}{dx}\left(\int_1^{x^3} \sin t^2\,dt - \int_1^{2x} \sin t^2\,dt\right)$$
$$= (3x^2)\sin(x^3)^2 - 2\sin(2x)^2 = 3x^2\sin x^6 - 2\sin 4x^2$$

by breaking up the domain of integration, using the fundamental theorem of calculus, and the chain rule.

(j)

$$\int_0^{\sqrt{\pi}} \frac{d}{dt}\cos t^2\,dt = \cos t^2\Big|_0^{\sqrt{\pi}} = \cos\pi - \cos 0 = -2$$

(k) Let $u = f(x)$ then $du = \frac{du}{dx}dx = f'(x)\,dx$. Therefore,

$$\int_2^4 f'(x)\sin(f(x))dx = \int_{f(2)}^{f(4)} \sin(u)\,du = \int_1^7 \sin(u)\,du = -\cos(7) + \cos(1).$$

(l) Notice that

$$\int (2x-8)e^{-x}\,dx = 2\int x\,e^{-x}\,dx - 8\int e^{-x}\,dx = 2\int x\,e^{-x}\,dx + 8e^{-x}$$

but by integration by parts,

$$\int x\,e^{-x}\,dx = -e^{-x} - xe^{-x}C$$

therefore,

$$\int (2x-8)e^{-x}\,dx = 6e^{-x} - 2xe^{-x} + C$$

and

$$\int_1^3 (2x-8)e^{-x}\,dx = -\frac{4}{e}$$

(m)

$$\int_{-1}^{10} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_0^3 f(x)\,dx + \int_3^{10} f(x)\,dx$$
$$= \int_{-1}^{0} (-x^2)\,dx + \int_0^3 (2x)\,dx + \int_3^{10} (-5)\,dx = -\frac{1}{3} + 9 - 35 = -\frac{79}{3}.$$

2. John moves along the real line with velocity at time t given by $v(t) = \sin^3 t$ and his position at time t is denoted by $s(t)$. Furthermore at $t = 0$, John is standing at $s(0) = 5$.
(a) Find John's position at $t = \frac{5}{4}\pi$. Since $v(t) = s'(t)$,

$$s(t) = \int \sin^3 t\,dt = \int (1 - \cos^2 t)\sin t\,dt = \int \sin t\,dt - \int \cos^2 t\,\sin t\,dt$$
$$= -\cos t + \frac{1}{3}\cos^3 t + C.$$

Since $5 = s(0) = -\cos 0 + \frac{1}{3}\cos^3 0 + C$, $C = \frac{17}{3}$ so $s(t) = -\cos t + \frac{1}{3}\cos^3 t + \frac{17}{3}$. Therefore,

$$s\left(\frac{5}{4}\pi\right) = \frac{5}{6\sqrt{2}} + \frac{17}{3}.$$

(b) Find John's displacement between times $t = 0$ and $t = \frac{5}{4}\pi$.

$$s\left(\frac{5}{4}\pi\right) - s(0) = \frac{5}{6\sqrt{2}} + \frac{2}{3}.$$

(c) Find the total distance traveled by John between times $t = 0$ and $t = \frac{5}{4}\pi$.

$$\int_0^{\frac{5}{4}\pi} |\sin^3 t|\,dt = \int_0^{\pi} \sin^3 t\,dt + \int_{\pi}^{\frac{5}{4}\pi} (-\sin^3 t)\,dt = \frac{4}{3} + (\frac{2}{3} - \frac{5}{6\sqrt{2}}) = 2 - \frac{5}{6\sqrt{2}}.$$

3. Consider the following Riemann sum:

$$I := \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^8 \left(\frac{2}{n}\right).$$

(a) Write I as a definite integral.

$$\int_1^3 x^8\,dx$$

(b) Calculate I (using any method you like).

$$\int_1^3 x^8\,dx = \frac{3^9}{9} - \frac{1^9}{9} = \frac{19682}{9}$$

4. Calculate the area of the region bounded by the x-axis, and $y = -x^2 + 4x + 5$ which is to the left of the line $x = \pi$.

Notice that $y = -x^2 + 4x + 5 = (5 - x)(1 + x)$ which crosses the x-axis when $x = 5, -1$. Therefore, the desired area is

$$\int_{-1}^{\pi} (-x^2 + 4x + 5)\,dx = \frac{8}{3} + 5\,\pi + 2\,\pi^2 - \frac{\pi^3}{3}.$$

5. Consider the graph below. Find the following:

(a)

$$\int_{-5}^{0} f(x)\,dx = \int_{-5}^{-4} f(x)\,dx + \int_{-4}^{-3} f(x)\,dx + \int_{-3}^{0} f(x)\,dx = (3)(1) + (-4)(1) - \frac{1}{2}(4)(3) = -7.$$

(b)

$$F'(x) = \frac{d}{dx}\int_{-3}^{x^4} f(t)\,dt = f(x^4)\frac{d}{dx}(x^4) = 4x^3 f(x^4).$$

Therefore, $F'(-1) = 4(-1)f(1) = -4(2) = -8$.

(c)

$$\int_{-3}^{1} |\,f(x)\,|\,dx = \int_{-3}^{0} |\,f(x)\,|\,dx + \int_{0}^{1} |\,f(x)\,|\,dx = \frac{1}{2}(3)(4) + \frac{1}{2}(1)(2) = 7$$

(d)

$$\int_{-3}^{4} f'(x)\,dx = f(4) - f(-3) = -1 - (-4) = 3.$$

(e) Let $u = x^2$ then $du = \frac{du}{dx}\,dx = 2x dx$. Furthermore, $(-1)^2 = 1$ and $2^2 = 4$ thus,

$$\int_{-1}^{2} f'(x^2)\,x\,dx = \int_{1}^{4} f'(u)\,\frac{1}{2}\,du = \frac{1}{2}(f(4) - f(1)) = \frac{1}{2}(-1 - 2) = -\frac{3}{2}.$$

(f)

$$\int_5^7 (9(f(x))^2 - 8)\,dx = \int_5^7 (9(2)^2 - 8)\,dx = 56$$