SPIN GROMOV-WITTEN INVARIANTS

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Abstract. We define and study r-spin Gromov-Witten invariants and r-spin quantum cohomology of a projective variety V, where $r \geq 2$ is an integer. The main element of the construction is the space $\overline{\mathcal{M}}_{g,n}^{1/r}(V)$ of r-spin maps, the stable maps into a variety V from n-pointed algebraic curves of genus g with the additional data of an r-spin structure on the curve. We prove that $\overline{\mathcal{M}}_{g,n}^{1/r}(V)$ is a Deligne-Mumford stack and use it to define the r-spin Gromov-Witten classes of V. We show that these classes yield a cohomological field theory (CohFT) which is isomorphic to the tensor product of the CohFT associated to the usual Gromov-Witten invariants of V and the r-spin CohFT. Restricting to genus zero, we obtain the notion of an r-spin quantum cohomology of V, whose Frobenius structure is isomorphic to the tensor product of the Frobenius manifolds corresponding to the quantum cohomology of V and the r-th Gelfand-Dickey hierarchy (or, equivalently, the A_{r-1} singularity). We also prove a generalization of the descent property which, in particular, explains the appearance of the ψ classes in the definition of gravitational descendants.

0. Introduction

In this paper, we present a generalization of the theory of quantum cohomology and Gromov-Witten invariants arising from algebraic curves with higher spin structures. Recall that the construction of the ordinary Gromov-Witten invariants of a projective variety V is based on the moduli spaces $\overline{\mathcal{M}}_{g,n}(V)$ of stable maps to V. The space $\overline{\mathcal{M}}_{g,n}(V)$ is a Deligne-Mumford stack compactifying the space of holomorphic maps to V from Riemann surfaces of genus g with n marked points. In particular, the moduli space of stable maps to a point coincides with the moduli of stable curves $\overline{\mathcal{M}}_{g,n}$.

Although the space $\overline{\mathcal{M}}_{g,n}(V)$ is not smooth in general, it has a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{virt}}$ which plays the role of the usual fundamental class in intersection theory. It gives rise to the collection of Gromov-Witten classes

$$\Lambda_{g,n}^{V} \in H^{\bullet}(\overline{\mathcal{M}}_{g,n}) \otimes \left(H^{\bullet}(V)^{\otimes n}\right)^{*},$$

defined as

$$\Lambda_{g,n}^{V}(\gamma_1, \ldots, \gamma_n) := \mathrm{st}_{*}\left(ev_1^{*}\gamma_1 \ldots ev_n^{*}\gamma_n \cap [\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{virt}}\right),$$

where $\gamma_j \in H^{\bullet}(V)$, $\mathrm{st} : \overline{\mathcal{M}}_{g,n}(V) \to \overline{\mathcal{M}}_{g,n}$ is the stabilization (forgetting the target) map, and $ev_j : \overline{\mathcal{M}}_{g,n}(V) \to V$ is the evaluation of the stable map at the j-th marked point. These classes behave nicely when restricted to the boundary

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strata of $\overline{\mathcal{M}}_{g,n}$. This allows one to define a collection of multilinear operations on the space $H^\bullet(V)$, parametrized by elements of $H_\bullet(\overline{\mathcal{M}}_{g,n})$. These operations satisfy the axioms of a cohomological field theory (CohFT) in the sense of Kontsevich-Manin [19]. In particular, their restriction to stable maps of genus zero endows $H^\bullet(V)$ with the structure of a (formal) Frobenius manifold [7, 10, 22], called the quantum cohomology of V, whose multiplication is a deformation of the usual cup product in $H^\bullet(V)$.

The diagonal map $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g,n}$ induces an operation of a tensor product in the category of CohFTs. Behrend [4] proved that despite the fact that the space $\overline{\mathcal{M}}_{g,n}(V \times V')$ is not isomorphic to the product $\overline{\mathcal{M}}_{g,n}(V) \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n}(V')$, the tensor product of the CohFTs associated to $\overline{\mathcal{M}}_{g,n}(V)$ and $\overline{\mathcal{M}}_{g,n}(V')$ is the CohFT associated to $\overline{\mathcal{M}}_{g,n}(V \times V')$. Restricting to genus zero, this gives a Künneth formula for quantum cohomology.

In [16], we introduced a new class of CohFTs, one for each integer $r \geq 2$, based on the moduli space $\overline{\mathcal{M}}^{1/r}_{g,n} = \coprod_{\mathbf{m}} \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ of higher spin curves, constructed in [11]. Recall that for $\mathbf{m} = (m_1, \dots, m_n)$, with $m_i \in \mathbb{Z}$, the moduli space $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ is a compactification of the space of Riemann surfaces of genus g with n marked points $p_1, \dots, p_n$ and an r-th root of the twisted canonical line bundle $\omega \otimes \mathcal{O}(-\sum_i m_i p_i)$. This CohFT has rank $r-1$ and is called an r-spin CohFT. The construction of an r-spin CohFT in [16] is based on a choice of a special cohomology class $c^{1/r}$ (called a spin virtual class) in $H^\bullet(\overline{\mathcal{M}}^{1/r}_{g,n})$, satisfying certain axioms. These axioms are similar to the Behrend-Manin axioms [5] for the virtual fundamental class. As in the case of the CohFT based on ordinary stable maps, the r-spin CohFT *a priori* may depend on a choice of the spin virtual class $c^{1/r}$. Currently, two different constructions of a spin virtual class on $\overline{\mathcal{M}}^{1/r}_{g,n}$ are known: an algebro-geometric construction of [27], resembling the algebraic construction of the virtual fundamental class, and the analytic construction of [24] based on Witten's original idea [29]. While it is not known yet whether these constructions give the same class for all g and r, they agree when $g = 0$ (and any r) or $r = 2$ (and any g).

This r-spin CohFT is related to the work of Witten [29], who conjectured that a generating function of certain intersection numbers on $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ is a τ function of the r-th Gelfand-Dickey (or KdV_r) hierarchy. When $r = 2$, this conjecture reduces to an earlier conjecture of Witten's on the intersection numbers of $\overline{\mathcal{M}}_{g,n}$, which was proved by Kontsevich [18]. In [16], following Witten's ideas, we constructed the spin virtual class and proved the conjecture in the cases $g = 0$ (for all r) and $r = 2$ (for all g).

In [17], it was proven that the tensor product of the CohFTs associated to $\overline{\mathcal{M}}^{1/r}_{g,n}$ and $\overline{\mathcal{M}}^{1/r'}_{g,n}$ is realized by the moduli stack of curves endowed with both an r and an r' spin structure. More generally, moduli stacks of stable curves endowed with multiple spin structures provide an intersection-theoretic realization of the tensor products of spin CohFTs.

The goal of this paper is to complete this picture by introducing and studying moduli spaces that give an intersection-theoretic realization of the tensor product of the Gromov-Witten CohFT and the r-spin CohFT.

We construct $\overline{\mathcal{M}}^{1/r}_{g,n}(V)$, the stack of stable r-spin maps into a projective variety V—objects which combine both the data of a stable map and an r-spin structure. We prove that $\overline{\mathcal{M}}^{1/r}_{g,n}(V)$ is a Deligne-Mumford stack and a ramified cover of $\overline{\mathcal{M}}_{g,n}(V)$. Similar to the case of ordinary stable maps, stable r-spin maps to a point are just stable r-spin curves.

We introduce the spin virtual class $\tilde{c}^{1/r} \in H^\bullet(\overline{\mathcal{M}}^{1/r}_{g,n}(V))$, which is an analog of a class $c^{1/r}$ on $\overline{\mathcal{M}}^{1/r}_{g,n}$. Using the class $\tilde{c}^{1/r}$ and the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(V)$, we define the r-spin Gromov-Witten classes

$$\Lambda^{(V,r)}_{g,n} \in H^\bullet(\overline{\mathcal{M}}^{1/r}_{g,n}(V)) \otimes \left(\mathcal{H}^{(V,r)^{\otimes n}}\right)^*,$$

where $\mathcal{H}^{(V,r)} = H^\bullet(V) \otimes \mathcal{H}^{(r)}$, and $\mathcal{H}^{(r)}$ is the state space of the r-spin CohFT. We prove that these spin Gromov-Witten classes give rise to a CohFT with the state space $\mathcal{H}^{(V,r)}$, which is isomorphic to the tensor product of the Gromov-Witten CohFT and the r-spin CohFT. As with Behrend's theorem, this result is not trivial because the space $\overline{\mathcal{M}}^{1/r}_{g,n}(V)$ is not isomorphic to the fiber product $\overline{\mathcal{M}}_{g,n}(V) \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}^{1/r}_{g,n}$. Restricting to genus zero, we obtain that the r-spin quantum cohomology of V is the tensor product of its ordinary quantum cohomology with the Frobenius manifold associated to KdV_r (or equivalently, to the A_{r-1} singularity).

It is worth observing that our spin Gromov-Witten invariants have a physical interpretation. They may be regarded as the correlators in a theory of topological gravity coupled to topological matter, where the matter sector of the theory is the topological sigma model with target space V coupled with a certain type of gauged $SU(2)_{r-2}/U(1)$ Wess-Zumino-Witten model. It would be very interesting to find an enumerative interpretation of the these invariants similar to the interpretation of the ordinary Gromov-Witten invariants.

Structure of the paper. We will now give a more detailed description of the structure of the paper and of our results.

After a brief review in the first section of the ideas of r-spin structures (in order to set notation that will be necessary thereafter), we begin in the second section by setting up the geometric framework for the rest of the paper. We introduce stable spin maps and the stack $\overline{\mathcal{M}}^{1/r}_{g,n}(V)$ of such maps. We prove that it is a Deligne-Mumford stack, and we establish important properties of its associated morphisms. The proof of Theorem 2.2.1, that the stabilization map is truly a morphism of stacks, is rather involved and concludes the second section.

In the third section, we introduce cohomology classes on $\overline{\mathcal{M}}^{1/r}_{g,n}(V)$, especially the spin virtual class $\tilde{c}^{1/r}$ on $\overline{\mathcal{M}}^{1/r}_{g,n}(V)$. The class $\tilde{c}^{1/r}$ is defined by pulling back $c^{1/r}$ from $\overline{\mathcal{M}}^{1/r}_{g,n}$ when $2g-2+n>0$, and it is defined by a direct construction for other values of g and n. We establish some properties of the spin virtual class and relate it, in genus zero, to the top Chern class of the top cohomology $R^1\pi_*\mathcal{E}_r$ of the r-spin structure bundle. At the end of this section we prove a key theorem (Theorem 3.3.1) on the decomposition of classes pushed down from $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)$ to $\overline{\mathcal{M}}_{g,n}$. This theorem is the main ingredient in proving Theorem 4.3.2, that the CohFT arising from r-spin maps is a tensor product.

In the fourth section, using the class $\tilde{c}^{1/r}$ and the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(V)$, we define the r-spin Gromov-Witten classes $\Lambda_{g,n}^{(V,r)}$. We show that these classes give rise to a CohFT which has state space $\mathcal{H}^{(V,r)}$, and which is isomorphic (in the stable range) to the tensor product of the Gromov-Witten CohFT and the r-spin CohFT. We prove that these classes satisfy properties analogous to Gromov-Witten classes, and we show that even in the unstable range, at least for the small phase space, all the correlators of the stable-spin-maps CohFT are determined by the usual r-spin CohFT and the Gromov-Witten invariants. Finally, we verify that the descent axiom of [15] holds for these correlators in the genus zero case. This gives a new and interesting geometric description of gravitational descendants not only in the r-spin theory, but also in the case of usual Gromov-Witten invariants, since these correspond to the special case of $r = 2$, as described in Section 5.

In the fifth and last section, we discuss a number of special cases. We first prove that when $r = 2$, the stable spin maps CohFT reduces to the usual Gromov-Witten invariants of V. We then examine the case of genus zero and degree zero, and conclude with the calculation of the small phase space potential function associated to $\overline{\mathcal{M}}_{0,n}^{1/3}(\mathbb{P}^1)$.

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1. Review of spin structures

For the remainder of the paper we fix an integer $r \geq 2$. For the reader's convenience and to establish notation, we briefly review the definitions of an r-spin structure given in [11]. Our notation here is somewhat improved over that of [11].

1.1. **Overview.**

Although the concept of an r-spin structure is intuitively simple, its formal definition is somewhat technical. For that reason we first give a brief overview of the ideas involved.

Intuitively, an r-spin structure on a smooth, n-pointed curve $(X, p_1, \ldots, p_n)$ is just a choice of a line bundle $\mathcal{L}$ on X, together with an isomorphism

$$b : \mathcal{L}^{\otimes r} \longrightarrow \omega_X(-\sum m_i p_i)$$

to the canonical dualizing sheaf of X with zeros of order $\mathbf{m} = (m_1, \ldots, m_n)$, for some n-tuple of non-negative integers $\mathbf{m}$.

For degree reasons, an r-spin structure of type $\mathbf{m}$ exists on a genus g curve X only if $2g - 2 - \sum m_i$ is divisible by r.

If we want to compactify the spaces involved by considering stable maps and prestable curves, the preceding, intuitive definition of an r-spin structure is insufficient. In particular, we must replace line bundles by rank-one torsion-free sheaves and allow the homomorphism $b : \mathcal{L}^{\otimes r} \to \omega_X(-\sum m_i p_i)$ to have non-trivial cokernel at the nodes of the curve.

Alternatively, one may continue to use invertible sheaves, but allow the source curves to be stacks ("twisted" nodal curves, or orbicurves) as in [1]. The two approaches are completely equivalent and give isomorphic compactifications. Although the orbicurves approach is more attractive notationally, the approach based on torsion-free sheaves is more explicit and is closer to the original physically motivated constructions of [29]. Also, it is better suited to the treatment of the descent axiom, a generalization of which we prove in Subsection 4.6. This generalized descent property gives a nice geometric explanation for the appearance of ψ classes in the usual (non-spin) Gromov-Witten theory. Therefore, we feel that the approach based on torsion-free sheaves is more suitable for the purposes of this paper.

There are two very different types of behavior of the torsion-free sheaf $\mathcal{L}$ near a node $q \in X$. When it is still locally free, the sheaf $\mathcal{L}$ is said to be *Ramond* at the node q. If the sheaf $\mathcal{L}$ is not locally free at q, it is called *Neveu-Schwarz*. (In the twisted curve formulation, the Ramond case corresponds to a trivial stack structure at the point in question, while the Neveu-Schwarz case corresponds to a non-trivial stack structure at that point.)

Although in the Ramond case, the homomorphism b remains an isomorphism near the node q, in the Neveu-Schwarz case it cannot be an isomorphism. The local structure of the sheaf $\mathcal{L}$ near a Neveu-Schwarz node can be described as follows.

Near the node q, the structure sheaf $\mathcal{O}_X$ is generated by two functions x and y, such that $xy = 0$. The sheaf $\omega_X(-\sum m_i p_i)$ is locally generated by $\frac{dx}{x} = -\frac{dy}{y}$. Near q the sheaf $\mathcal{L}$ is generated by two elements ℓ_+ and ℓ_-, supported on the x and y branches respectively (that is, $x\ell_- = y\ell_+ = 0$). The two generators may be chosen so that the homomorphism $b : \mathcal{L}^{\otimes r} \to \omega_X(-\sum m_i p_i)$ takes $\ell_+^{\otimes r}$ to $x^{m_+ +1}(\frac{dx}{x}) = x^{m_+}dx$ and $\ell_-^{\otimes r}$ to $y^{m_- +1}(\frac{dy}{y}) = y^{m_-}dy$, where $(m_+ + 1) + (m_- + 1) = r$.

Definition 1.1.1. We call m_+ (respectively m_-) the *order of the spin structure along the x-branch (respectively y-branch).*

1.2. **Formal definitions.**

Definition 1.2.1. For any integer $r > 1$ and for any n-tuple of integers $\mathbf{m} = (m_1, \ldots, m_n)$ such that r divides $2g - 2 - \sum m_i$, an *r-spin structure of type* $\mathbf{m}$ on a family X/T of n-pointed prestable curves is a coherent net of r-th roots of $\omega_{X/T}$ of type $\mathbf{m}$.

Recall that, by Definition 2.3.4 of [11], a *coherent net of r-th roots* of $\omega_{X/T}$ of type $\mathbf{m}$ is a set of rank-one, torsion-free $\mathcal{O}_X$-modules $\{\mathcal{E}_d\}$ for every positive $d|r$, and a collection of $\mathcal{O}_X$-module homomorphisms $\{c_{d,d'} : \mathcal{E}_d^{\otimes d/d'} \to \mathcal{E}_{d'}\}$, defined for every positive $d'|d|r$, such that for each geometric fiber X_t of X/T, the sheaves $\{\mathcal{E}_d\}$ and homomorphisms $\{c_{d,d'}\}$ induce a coherent net of r-th roots of ω_{X_t} of type $\mathbf{m}$, and each homomorphism $c_{d,d'}$ is an isomorphism on the locus in X/T where $\mathcal{E}_d$ is locally free. That is,

- $\mathcal{E}_1 = \omega_{X_t}$, and $c_{d,d} = \mathbf{1}_d$ is the identity map, for every positive d dividing r.
- For each divisor d of r and each divisor d' of d, we require:
  - For every point $p \in X_t$ where $\mathcal{E}_d$ is not free, the length of the cokernel of $c_{d,d'}$ at p is $(d/d') - 1$.
  - $d \cdot \deg \mathcal{E}_d = d' \cdot (\deg \mathcal{E}_{d'} - \sum m'_i)$, where $\mathbf{m}' = (m'_1, \ldots, m'_n)$ is the reduction of $\mathbf{m}$ modulo d/d' (i.e. $0 \le m'_i < d/d'$ and $m_i \equiv m'_i \pmod{d/d'}$).

- The homomorphisms $\{c_{d,d'}\}$ are compatible. That is, the diagram

$$\begin{tikzcd}
(\mathcal{E}_d^{\otimes d/d'})^{\otimes d'/d''} \arrow[r, "(c_{d,d'})^{\otimes d'/d''}"] \arrow[dr, "c_{d,d''}"'] & \mathcal{E}_{d'}^{\otimes d'/d''} \arrow[d, "c_{d',d''}"] \\
 & \mathcal{E}_{d''}
\end{tikzcd}$$

  commutes for every $d''|d'|d|r$.

Finally, recall that these sheaves and homomorphisms must have a special type of local structure. The details of these conditions, while rather technical, are important for the proof of Theorem 2.2.1. We review them briefly here, for the reader's convenience and for purposes of fixing notation.

For a node $q \in X_t$ in a fiber of X/T over a geometric point $t \in T$, we denote by $m_{d,+}$ and $m_{d,-}$ the orders of the d-th root map

$$c_{d,1} : \mathcal{E}_d^{\otimes d} \to \omega(-\sum m_i p_i)$$

on the branches of the normalization of X_t at q. We define

$$u_d := (m_{d,+} + 1)/\ell_d \quad \text{and} \quad v_d := (m_{d,-} + 1)/\ell_d,$$

where

$$\ell_d := \gcd(m_{d,+} + 1, m_{d,-} + 1).$$

If $c_{d,1}$ is an isomorphism at q, we set $u_d = v_d = 0$.

The first requirement on the local structure of a net of coherent roots on a family X/T is the existence of a special local coordinate system near any node q where $c_{r,1}$ is not an isomorphism (i.e., $\mathcal{E}_r$ is Neveu-Schwarz at q). This local coordinate system consists of an étale neighborhood T' of t with an element $\tau \in \mathcal{O}_{T',t}$, and an étale neighborhood U of q in $X \times_T T'$ with sections $x, y \in \mathcal{O}_U$, such that for $s := u_r + v_r$ we have

- $xy = \tau^s$.
- The ideal generated by x and y has the singular locus of X/T as its associated closed subscheme.
- The homomorphism $\big(\mathcal{O}_{T',t}[x,y]/(xy-\tau^s)\big) \to \mathcal{O}_{U,q}$ induces an isomorphism of the completions

$$\left(\hat{\mathcal{O}}_{T',t}[[x,y]]/(xy - \tau^s)\right) \xrightarrow{\sim} \hat{\mathcal{O}}_{U,q}.$$

The second requirement on the local structure is that the sheaves $\mathcal{E}_d$ must have a special presentation in terms of this special coordinate system. In particular, any rank-one, torsion-free sheaf $\mathcal{F}$ always has a presentation of the form

$$\mathcal{F} \cong \langle \zeta_1, \zeta_2 | e\zeta_1 = x\zeta_2,\ y\zeta_1 = h\zeta_2 \rangle$$

for some e and h in $\mathcal{O}_{T',t}$, such that $eh = \tau^s$; but for sheaves in the net we require that if $\mathcal{E}_d$ is not locally free at the node q, then $\mathcal{E}_d$ must have such a presentation with $e = \tau^{(r/d)(v_d\ell_d)}$ and $h = \tau^{(r/d)(u_d\ell_d)}$. In other words, $\mathcal{E}_d$ is isomorphic near the node q to the sheaf

$$E_d := \langle \zeta_1, \zeta_2 | \tau^{(r/d)(v_d\ell_d)}\zeta_1 = x\zeta_2,\ y\zeta_1 = \tau^{(r/d)(u_d\ell_d)}\zeta_2 \rangle.$$

If $\mathcal{E}_d$ is locally free at q, then for uniformity of notation we will use the unusual presentation $\mathcal{E}_d \cong E_d := \langle \zeta_1, \zeta_2 | \zeta_1 = \zeta_2 \rangle$.

Finally, each homomorphism

$$c_{dj,j} : \mathcal{E}_{dj}^{\otimes d} \longrightarrow \mathcal{E}_j$$

in the net must be a so-called *power map*, in the sense of Definition 2.3.1 of [11]. This means that, if we use the local presentations

$$E_{dj} = \langle \xi_1, \xi_2 | \tau^{(r/(dj))(v_{dj}\ell_{dj})}\xi_1 = x\xi_2,\ y\xi_1 = \tau^{(r/(dj))(u_{dj}\ell_{dj})}\xi_2 \rangle,$$

and

$$E_j = \langle \zeta_1, \zeta_2 | \tau^{(r/j)(v_j\ell_j)}\zeta_1 = x\zeta_2,\ y\zeta_1 = \tau^{(r/j)(u_j\ell_j)}\zeta_2 \rangle,$$

of the sheaves $\mathcal{E}_{dj}$ and $\mathcal{E}_j$, then the map

$$\mathbf{Sym}^d(E_{dj}) \to E_j, \tag{1}$$

induced by the homomorphism $c_{dj,j}$, acts on the generators $\xi_1^{d-i}\xi_2^i$ of $\mathbf{Sym}^d(E_{dj})$ as

$$\xi_1^{d-i}\xi_2^i \mapsto \begin{cases} x^{u''-i}\tau^{iv}\zeta_1 & \text{if } 0 \le i \le u'' \\ y^{v''-d+i}\tau^{(d-i)v}\zeta_2 & \text{if } u'' < i \le d. \end{cases} \tag{2}$$

Here we require that $u_j \equiv u_{dj}d \pmod{s}$ and $v_j \equiv v_{dj}d \pmod{s}$, and we define $u'' := (u_{dj}d - u_j)/s$ and $v'' := (v_{dj}d - v_j)/s$.

If $\mathcal{E}_{dj}$ is locally free at q, then the existence of a good presentation is automatically satisfied, and we have no additional power map requirement except that the map (1) be an isomorphism.

2. Stable spin maps

In this section we introduce stable r-spin maps and begin to study their moduli stack.

2.1. Definitions.

Definition 2.1.1. Let $r \ge 2$ be an integer, and let n and g be non-negative integers. Let V be an algebraic variety, and let β be a class in $H_2(V, \mathbb{Z})$. Finally, let $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ be an n-tuple of integers such that r divides $2g - 2 - \sum m_i$. A *family of stable, n-pointed, r-spin maps into V of genus g, type $\mathbf{m}$, and class β* is a pair $(f, (\{\mathcal{E}_d\}, \{c_{d,d'}\}))$, consisting of a family of stable n-pointed genus g maps $f : X/T \to V$ of class β, and an r-spin structure $(\{\mathcal{E}_d\}, \{c_{d,d'}\})$ of type $\mathbf{m}$ on X/T.

Example 2.1.2. If V is a point, then any stable, n-pointed, r-spin map into V is just a stable r-spin curve.

Definition 2.1.3. An isomorphism from an r-spin map $(X \xrightarrow{f} V, p_1, \ldots, p_n, (\{\mathcal{E}_d\}, \{c_{d,d'}\}))$ to another $(X' \xrightarrow{f'} V, p'_1, \ldots, p'_n, (\{\mathcal{E}'_d\}, \{c'_{d,d'}\}))$ of the same type $\mathbf{m}$ consists of an isomorphism τ of n-pointed, stable maps

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ V \arrow[r, equal] & V \end{tikzcd}$$

and a set of $\mathcal{O}_X$-module isomorphisms $\{\theta_d : \tau^*\mathcal{E}'_d \xrightarrow{\sim} \mathcal{E}_d\}$, with θ_1 being the canonical isomorphism $\tau^*\omega_{X'}(-\sum_i m_i {p'}_i) \xrightarrow{\sim} \omega_X(-\sum m_i p_i)$, and such that the homomorphisms θ_d are compatible with all the maps $c_{d,d'}$ and $\tau^* c'_{d,d'}$.

Definition 2.1.4. Let V be an algebraic variety over $\mathbb{C}$, and β an element of $H_2(V,\mathbb{Z})$. The *stack of stable r-spin maps to V* (n-pointed, of genus g, and class β) is the disjoint union

$$\overline{\mathcal{M}}^{1/r}_{g,n}(V,\beta) := \coprod_{\substack{\mathbf{m} \\ 0\le m_i<r}} \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)$$

of stacks $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)$ of (families of) stable n-pointed r-spin maps to V of genus g, type $\mathbf{m} = (m_1,\dots,m_n)$, and class β.

We will see in Section 2.3 that $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)$ (and, therefore, $\overline{\mathcal{M}}^{1/r}_{g,n}(V,\beta)$) is a Deligne-Mumford stack whenever $\overline{\mathcal{M}}_{g,n}(V,\beta)$ is. As in the special case of $V = pt$, no information is lost by restricting $\mathbf{m}$ to the range $0 \le m_i \le r-1$.

Proposition 2.1.5. *If* $\mathbf{m} \equiv \mathbf{m}' \pmod r$*, then* $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)$ *is canonically isomorphic to* $\overline{\mathcal{M}}^{1/r,\mathbf{m}'}_{g,n}(V,\beta)$.

Proof. When $\mathbf{m} \equiv \mathbf{m}' \pmod r$, every r-spin structure of type $\mathbf{m}$ naturally gives an r-spin structure of type $\mathbf{m}'$ simply by

$$\mathcal{E}_r \mapsto \mathcal{E}_r \otimes \mathcal{O}\left(\sum \frac{m_i - {m'}_i}{r} p_i\right)$$

□

2.2. **Fundamental morphisms of stacks of stable spin maps.**

The stack $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)$ has a natural projection

$$\tilde{p} : \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n}(V,\beta) \tag{3}$$

which forgets the spin structure. The usual evaluation maps

$$ev_i : \overline{\mathcal{M}}_{g,n}(V,\beta) \to V,$$

which send a point $[X \xrightarrow{f} V, p_1 \dots p_n] \in \overline{\mathcal{M}}_{g,n}(V,\beta)$ to $f(p_i) \in V$, induce evaluation maps

$$\tilde{ev}_i = ev_i \circ \tilde{p} : \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta) \longrightarrow V.$$

Less obvious is the fact that for any morphism $s : V \to V'$ taking β to β', we have a stabilization morphism

$$\tilde{st} : \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta) \longrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V',\beta') \tag{4}$$

which takes f to $f' := s \circ f$ and contracts components of the source curve that are unstable with respect to f'.

Theorem 2.2.1. *For any morphism* $V \to V'$*, taking* β *to* β'*, the stabilization map (4) is a morphism of stacks.*

The proof of Theorem 2.2.1, which is rather intricate, will be given in Subsection 2.4.

The various canonical maps introduced above are shown in the following commutative diagram.

$$
\begin{tikzcd}
& & \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta) \arrow[ddll, "\tilde{st}"'] \arrow[d, "q_1"] \arrow[ddrr, "\tilde{p}"] \arrow[ddrrr, "\tilde{ev}_i"] & & & \\
& & \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n}(V,\beta) \arrow[dll, "pr_1"'] \arrow[drr, "pr_2"] \arrow[dd, "q_2"] & & & \\
\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \arrow[drr, "p"'] & & & & \overline{\mathcal{M}}_{g,n}(V,\beta) \arrow[r, "ev_i"] \arrow[dll, "st"] & V \\
& & \overline{\mathcal{M}}_{g,n} & & &
\end{tikzcd}
\tag{5}
$$

We will use the notation of this diagram throughout the remainder of the paper, and we will denote the composition $q_2 \circ q_1$ by q.

The universal curves $\mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ and $\mathcal{C}^{1/r,\mathbf{m}}_{g,n} \to \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)$ will be denoted by π.

Remark 2.2.2. The stack $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)$ is not isomorphic to the fibered product $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n}(V,\beta)$, although on the smooth locus the map

$$
q_1 : \mathcal{M}^{1/r,\mathbf{m}}_{g,n}(V,\beta) \longrightarrow \mathcal{M}^{1/r,\mathbf{m}}_{g,n} \times_{\mathcal{M}_{g,n}} \mathcal{M}_{g,n}(V,\beta)
$$

is an isomorphism when g and n are in the stable range $(2g-2+n>0)$.

The isomorphism for the smooth locus is straightforward: If X/T is a smooth family of curves, then a stable map $f : X \to V$ and an r-spin structure $(\{\mathcal{E}_d\}, \{c_{d,d'}\})$ are precisely the data necessary to construct an r-spin map, i.e., there is a canonical morphism

$$
j : \mathcal{M}^{1/r,\mathbf{m}}_{g,n} \times_{\mathcal{M}_{g,n}} \mathcal{M}_{g,n}(V,\beta) \to \mathcal{M}^{1/r,\mathbf{m}}_{g,n}(V,\beta)
$$

which is clearly the inverse of the morphism q_1.

But when the curve X is not stable, this morphism j no longer exists. For example, let X be a prestable curve that has two irreducible components C and E, where C is a smooth curve of genus g, and E is a smooth, rational curve, without marked points, joined to C at a single node $\mathfrak{q}$. Let $f : X \to V$ be an embedding of X in V.

An r-spin structure $(\{\mathcal{E}_d\}, \{c_{d,d'}\})$ on X is equivalent to a pair of r-spin structures $(\{\mathcal{E}'_d\}, \{c'_{d,d'}\})$ on C and $(\{\mathcal{E}''_d\}, \{c''_{d,d'}\})$ on E of orders 0 and $r-2$, respectively, at $\mathfrak{q}$. Thus the automorphism group of the r-spin map $(f, (\{\mathcal{E}_d\}, \{c_{d,d'}\}))$ is $\mu_r \times \mu_r$, corresponding to multiplication of $\mathcal{E}'_r$ and $\mathcal{E}''_r$ by r-th roots of unity. But the stabilization map $\tilde{st}$ takes $(f, (\{\mathcal{E}_d\}, \{c_{d,d'}\}))$ to the spin map $(f|_C, \{\mathcal{E}'_d\}, \{c'_{d,d'}\})$ on

C, and the automorphism group of

$$\tilde{p}(f, (\{\mathcal{E}_d\}, \{c_{d,d'}\})) \times \tilde{st}(f, (\{\mathcal{E}_d\}, \{c_{d,d'}\})) = (f|_C, (\{\mathcal{E}'_d\}), \{c'_{d,d'}\})$$

is simply μ_r, since C is irreducible and $\mathcal{E}'_r$ is invertible on C. Thus the morphism q_1 is not an isomorphism.

Proposition 2.2.3. *The morphism q_1 is flat and proper.*

Proof. Flatness follows from the valuative criterion of flatness [8, 11.8.1], which states that it is enough to check flatness of q_1 over each R-valued point

$$\operatorname{Spec} R \to \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n}(V,\beta),$$

where R is a discrete valuation ring. Since the completion $\hat{R}$ of R is faithfully flat over R, it suffices to check this for each complete discrete valuation ring. But in this case, the results of [11] show that the universal deformation (relative to the universal stable map $f : \mathcal{C} \to V$) of a spin structure over the central fiber of $\operatorname{Spec} R$ corresponds to the ring homomorphism $R \to R[t]/(t^d - s)$, for some positive d dividing r, and where $s \in R$ is a uniformizing parameter for R. In particular, $R[t]/(t^d - s)$ is a free R-module, and thus is flat over R. Since the universal deformation is faithfully flat (actually, étale) over $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)$, this shows that q_1 is also flat.

Properness also follows by the valuative criterion in exactly the same manner as was proved in [12] for spin structures on stable curves. Nothing in that proof required the underlying curves to be stable—only prestable. □

2.3. **The algebraic nature of the stack of stable spin maps.**

A useful notion in dealing with stacks is the idea of a Deligne-Mumford morphism, or morphism of Deligne-Mumford type. This is analogous to the concept of a representable morphism.

Definition 2.3.1. A morphism of stacks $f : S \to T$ is called *Deligne-Mumford* (or *of Deligne-Mumford type*) if for every representable U and every U-valued point $U \to T$, the fibered product $S \times_T U$ is a Deligne-Mumford stack.

The most useful fact about these morphisms is that if $S \to T$ is a Deligne-Mumford morphism, and if T is a Deligne-Mumford stack, then S is a Deligne-Mumford stack (see [14, Prop. 3.1.3]).

Theorem 2.3.2. *For all V and β, the forgetful morphism (equation (3)) is a finite (meaning proper and quasi-finite, but not necessarily representable) Deligne-Mumford morphism of stacks. In particular, $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)$ is a Deligne-Mumford stack whenever $\overline{\mathcal{M}}_{g,n}(V,\beta)$ is.*

Proof. Given a T-valued point $T \to \overline{\mathcal{M}}_{g,n}(V,\beta)$ for a representable T, we must show that the stack

$$R(X/T) := \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta) \times_{\overline{\mathcal{M}}_{g,n}(V,\beta)} T$$

of coherent nets of r-th roots of $\omega_X(-\sum m_i p_i)$ on the associated family X/T of prestable curves is a Deligne-Mumford stack, finite over T. In particular, we need to construct a smooth cover of $R(X/T)$ and show that the diagonal

$$\Delta : R(X/T) \times_T R(X/T) \longrightarrow R(X/T)$$

is representable, unramified, and proper.

These facts are all straightforward generalizations of their counterparts over the stack $\overline{\mathcal{M}}_{g,n}$ of stable curves as described in [11]. The only real difference is that we are now working with a specific family of prestable curves over T, as opposed to working with the universal family of stable curves (over $\overline{\mathcal{M}}_{g,n}$), but that changes nothing of substance in the proof.

The proof of properness of $R(X/T) \longrightarrow T$ is also an easy generalization of the case of stable r-spin curves, and the morphism is obviously quasi-finite. □

2.4. **Proof that stabilization is a morphism.**

We now turn to the proof of Theorem 2.2.1, that for any morphism $s : V \to V'$, taking $\beta \in H_2(V, \mathbb{Z})$ to $\beta' \in H_2(V', \mathbb{Z})$, the stabilization map $\tilde{st}$ (4) is a morphism of stacks.

It is straightforward to check that the stabilization of the underlying curves preserves r-spin structures on each individual fiber, but we must also show that the stabilization morphism on the underlying curves preserves the r-spin structure in families.

Theorem 2.2.1 obviously follows from the following lemma.

Lemma 2.4.1. *Let $st : \tilde{X}/T \to X/T$ be a morphism taking a family of n-pointed prestable curves $\tilde{X}/T$ to an n-pointed partial stabilization X of $\tilde{X}$, and let $(\{\tilde{\mathcal{E}}_d\}, \{\tilde{c}_{d,d'}\})$ be an r-spin structure of type $\mathbf{m} = (m_1, \ldots, m_n)$ on $\tilde{X}$, with $0 \leq m_i \leq r-1$ for every i. In this case, the sheaf $R^1 st_* \tilde{\mathcal{E}}_d$ is zero for every $d|r$, and the push-forward $(\{st_* \tilde{\mathcal{E}}_d\}, \{st_* \tilde{c}_{d,d'}\})$ is an r-spin structure of type $\mathbf{m}$ on X.*

Proof. As mentioned above, it is straightforward to check that the maps $st_* \tilde{c}_{d,d'}$ and the sheaves $st_* \tilde{\mathcal{E}}_d$ are T-flat and produce an r-spin structure of type $\mathbf{m}$ on each fiber of X/T (this will also follow from the computations below). Thus we only need to verify that $R^1 st_* \tilde{\mathcal{E}} = 0$ (which implies that this construction commutes with base change), and that the maps and sheaves meet the local conditions outlined in Subsection 1.2 for being a coherent net on the family of curves X/T, provided the original sheaves $\{\tilde{\mathcal{E}}_d\}$ and maps $\{\tilde{c}_{d,d'}\}$ form a coherent net on the family $\tilde{X}/T$.

Let us fix a point p of a geometric fiber X_t of X/T. There are three cases to consider. First is the case when the point p is not the image of a contracted component (i.e., $st^{-1}(p)$ is a single point). Second is the case when p is a smooth point of the fiber X_t, but p is the image of a whole irreducible component of the fiber $\tilde{X}_t$ of $\tilde{X}/T$; that is, st contracts a -1-curve to the point p. Third is the case that p is a node of the fiber X_t containing it, and it is the image of a contracted component of $\tilde{X}_t$; that is, p is the image of a -2-curve $\tilde{E}$.

Case 1: The first case is easy, since when $st^{-1}(p)$ is a single point, then st is an isomorphism in a neighborhood of p (or of $st^{-1}(p)$). In particular, st_* is an isomorphism, $R^1 st_* \tilde{\mathcal{E}}_d = 0$, and $c_{d,d'} = st_*(\tilde{c}_{d,d'})$ is a d/d'-th power map near p.

The second and third cases are more involved. Before we attack them, we note that the conditions we must verify are local (and analytic) on the base T, so it suffices to check the result when T is affine and is the spectrum of a complete local ring R. Moreover, the conditions are analytic on X; that is, the conditions are all determined by restricting to the completion of the local ring of X near the point p. To simplify, we will make the calculations in the case of $d = r$, but all other values of d (dividing r) are similar.

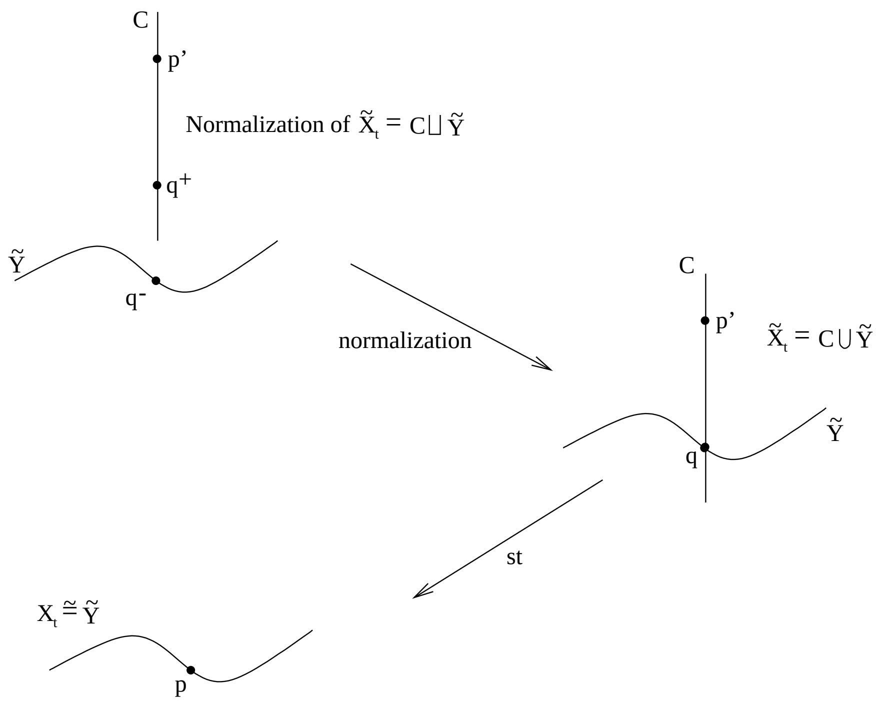


FIGURE 1. A depiction of Case 2 of Lemma 2.4.1: fibers $\tilde{X}_t$ and X_t, the stabilization map $st : \tilde{X}_t \to X_t$, and the normalization of $\tilde{X}_t$. The morphism st contracts the unstable component C to the point p and induces an isomorphism from the rest of the curve $\tilde{Y}$ to X_t.

Case 2: In the second case (st contracts a -1-curve of $\tilde{X}$ to the point p) we will show that the induced sheaves $st_*\tilde{\mathcal{E}}_d$ are locally free at p, and the maps $c_{d,d'}$ are all isomorphisms; thus the local coordinate and power map conditions are automatically fulfilled.

The fiber $\tilde{X}_t$ over X_t has one irreducible component C lying over p, and C contains at most one marked point p', labeled with an integer m, where $0 \leq m \leq r-1$. This is indicated in Figure 1. On C, the sheaf $(\tilde{\mathcal{E}}_r/\mathit{torsion})^{\otimes r}$ is isomorphic to $\omega_C(-m^+q^+ - mp)$, where q^+ is the point of C which maps to the node q attaching C to the rest of $\tilde{X}_t$.

Moreover, r must divide $2g_C - 2 - m^+ - m$, so either $m = r-1$, which implies that $\tilde{\mathcal{E}}_r$ is locally free (Ramond) near q, or $r - 2 = m^+ + m$, which implies that $\tilde{\mathcal{E}}_r$ is not locally free (it is Neveu-Schwarz) at q. In either case, $\tilde{\mathcal{E}}_r|_C$ has degree -1 and thus has no global sections. Also $R^1 st_*\tilde{\mathcal{E}}_r = 0$, since this is true on each fiber.

Now, in the Neveu-Schwarz case, the sheaf $st_*\tilde{\mathcal{E}}_r|_{X_t}$ is simply the sheaf $\tilde{\mathcal{E}}_r$ restricted (modulo torsion) to the rest of the prestable fiber $\tilde{Y} = \overline{(\tilde{X}_t - C)}$. But $\tilde{\mathcal{E}}_r/\mathit{torsion}$ on $\tilde{Y}$ is an r-th root of $\omega_{\tilde{Y}}(-m^-q^- - \sum_{p_i \neq p} m_i p_i)$, where q^- is the other side of the node defined by q^+. The actual value of m^- is determined by the relation $m^+ + m^- = r - 2$, which implies that $m^- = m$.

In the Ramond case, the vanishing of the global sections of $\tilde{\mathcal{E}}_r|_C$ implies that $st_*(\tilde{\mathcal{E}}_r|_{\tilde{X}_t})$ is $\tilde{\mathcal{E}}_r|\tilde{Y} \otimes \mathcal{E}(-q^-)$, so it is an r-th root of $\omega_{X_t}(-(r-1)p)$.

In both the Ramond and Neveu-Schwarz cases, the new marked point $p = st(q^-)$ of X_t is labeled with m, just as the old marked point p' was labeled with m on $\tilde{X}_t$. If no point was marked on C, then the point p remains unmarked (and $m^- = 0$).

Finally, $st_*\tilde{\mathcal{E}}_d$ is T-flat and $R^1 st_*\tilde{\mathcal{E}}_r$ vanishes, so we have that $st_*\tilde{\mathcal{E}}_r$ commutes with base change, and the calculations above on the fibers all hold globally on the family X/T. Thus $st_*\mathcal{E}_r$ is invertible near p, and $st_* c_{r,1}$ is an isomorphism near p. In particular, $st_* c_{r,1}$ is an r-power map. A similar argument holds for each $\tilde{\mathcal{E}}_d$ and each $\tilde{c}_{d,d'}$ near p.

Case 3: The third case is that of a point $p \in X$ which is the image of a -2-curve $\tilde{C}$ of $\tilde{X}$. Just as in Case 2, it is easy to see that on the unstable (contracted) -2-curve, the degree of the bundle is -1. Also, we have $R^1 st_*\tilde{\mathcal{E}}_r = 0$; the sheaf $st_*\tilde{\mathcal{E}}_r$ is T-flat and commutes with base change; and on the fibers, the induced collection of sheaves and bundles forms an r-spin structure of type $\mathbf{m}$.

We still must check that the induced sheaves have the necessary family structure for spin curves (existence of a local coordinate of suitable type, with respect to which the sheaves have the standard presentation—see Definition 1.2.1), and that the induced maps are power maps, as described in equation (1). For simplicity we will assume that the orders m_+, m_-, m'_+, and m'_- of the r-spin map $\tilde{c}_{r,1}$ along the two nodes q and q' where the -2-curve intersects the rest of the fiber have the property that $\gcd(m_+ + 1, m_- + 1) = 1 = \gcd(m'_+ + 1, m'_- + 1)$. The case with common divisors larger than 1 is similar.

It is shown in [12, §3.1] that $\tilde{X}$ is locally isomorphic to

$$\mathrm{Proj}_A\, A[\mu,\nu]/(x\nu - e^r\mu, h^r\nu - \mu y),$$

where $A = \hat{\mathcal{O}}_{X,x} \cong R[[x,y]]/xy - \pi^r$, and e, h and π are elements of the maximal ideal $\mathfrak{m}_R$ of R with $eh = \pi$. This shows the existence of the special local coordinate.

We next show that $st_*\mathcal{E}_d$ has a presentation of the form

$$st_*\mathcal{E}_d \cong \langle \zeta_1, \zeta_2 | \pi^{(r/d)(v_d\ell_d)}\zeta_1 = x\zeta_2,\ y\zeta_1 = \pi^{(r/d)(u_d\ell_d)}\zeta_2\rangle.$$

If we let $\mu/\nu = s$ and $\nu/\mu = z$, then near the exceptional -2-curve $\tilde{C}$ the curve $\tilde{X}$ is covered by two open sets,

$$U = \{\mu \neq 0\} \cong \mathrm{Spec}\, A[z]/(xz - e^r, y - h^r z)$$

and

$$V = \{\nu \neq 0\} \cong \mathrm{Spec}\, A[s]/(x - e^r s, ys - h^r).$$

Since $(\{\tilde{\mathcal{E}}_d\}, \{c_{d,d'}\})$ is an r-spin structure, we can describe $\tilde{\mathcal{E}}_r$ on U by $\tilde{\mathcal{E}}_r|_U \cong E_U(e^v, e^u) := \langle \zeta_1, \zeta_2 | z\zeta_2 = e^u\zeta_1,\ x\zeta_1 = e^v\zeta_2\rangle$, and on V by $\tilde{\mathcal{E}}_r|_V \cong E_V(h^{u'}, h^{v'}) := \langle \xi_1, \xi_2 | s\xi_2 = h^{u'}\xi_1,\ y\xi_1 = h^{v'}\xi_2\rangle$, where $u + v = u' + v' = r$.

On the exceptional curve $\tilde{C} \cong \mathbb{P}^1$, the sheaf $(\tilde{\mathcal{E}}_r/torsion)^{\otimes r}$ is isomorphic to $\omega_{\mathbb{P}^1}((1-u) + (1-v'))$, and degree considerations show that $u + v' = r$, so $u = u'$ and $v = v'$. Moreover, in a neighborhood of $\tilde{C}$, if D_i is the image of the i-th section $\mathfrak{p}_i : T \to X$, the invertible sheaf $\omega_{\tilde{X}}(-\sum m_i D_i)$ is trivial and is generated by the element $w = \frac{dx}{x} = -\frac{dz}{z} = \frac{ds}{s} = -\frac{dy}{y}$. The r-th power map $\tilde{c}_{r,1}$ is an isomorphism away from the nodes of $\tilde{X}$, and since it is a power map (changing the isomorphisms

$\tilde{\mathcal{E}}_r|_U \cong E_U(e^v, e^u)$ and $\tilde{\mathcal{E}}_r|_V \cong E_V(h^u, h^v)$, if necessary), it maps the generators ζ_i and ξ_i as follows:

$$\zeta_1^r \mapsto z^u w, \quad \zeta_2^r \mapsto x^v w$$

and

$$\xi_1^r \mapsto s^v w, \quad \xi_2^r \mapsto y^u w.$$

Since $\tilde{c}_{r,1}$ is an isomorphism away from the nodes, we have $\zeta_1^r = z^r \xi_1^r$, or $\zeta_1 = z\theta\xi_1$, for some r-th root of unity θ. Changing the isomorphism $\tilde{\mathcal{E}}_r|_V \cong E_V(h^u, h^v)$ by θ, we may assume

$$\zeta_1 = z\xi_1.$$

On $U \cap V$ we also have

$$\zeta_2 = se^v\zeta_1 = e^v\xi_1 \text{ and } \xi_2 = zh^u\xi_1 = h^u\zeta_1.$$

So global sections of $\tilde{\mathcal{E}}$ are of the form

$$\begin{aligned}\Gamma(\tilde{\mathcal{E}}_r) \quad = \quad & \{((f_U\zeta_1 + f'_U\zeta_2), (f_V\xi_1 + f'_V\xi_2)) \in E_U \oplus E_V | \\ & f_U\zeta_1 + f'_U\zeta_2 = f_V\xi_1 + f'_V\xi_2 \text{ on } U \cap V\}.\end{aligned}$$

We claim that the A-module

$$E(\pi^u, \pi^v) := \langle \eta_1, \eta_2 | x\eta_2 = \pi^u\eta_1, \ y\eta_1 = \pi^v\eta_2\rangle$$

is isomorphic to $\Gamma(\tilde{\mathcal{E}}_r)$ via

$$\eta_1 \mapsto (\zeta_2, e^v\xi_1) \text{ and } \eta_2 \mapsto (h^u\zeta_1, \xi_2).$$

The map is clearly an A-module homomorphism. Moreover, for any section $((f_U\zeta_1 + f'_U\zeta_2), (f_V\xi_1 + f'_V\xi_2)) \in \Gamma(\tilde{\mathcal{E}}_r)$ we may assume that $f_U \in R[z]$ and $f'_U \in R[[x]]$. Likewise, we may assume that $f_V \in R[s]$ and $f'_V \in R[[y]]$.

Consequently, we have

$$zf_U(z) + e^v f'_U(x) - f_V(s) - zq^u f'_V(y) = 0,$$

or

$$zf_U(z) + e^v f'_U(se^r) - f_V(s) - zh^u f'_V(zh^r) = 0.$$

Thus f_U and f_V are completely determined by

$$f_U = h^u f'_V(y) \text{ and } f_V = e^v f'_U(x).$$

We may, therefore, map $\Gamma(\tilde{\mathcal{E}}_r)$ to $E(\pi^u, \pi^v)$ via

$$(h^u f'_V(y)\zeta_1 + f'_U(x)\zeta_2), (e^v f'_U(x)\xi_1 + f'_V(y)\xi_2) \mapsto f'_U(x)\eta_1 + f'_V(y)\eta_2,$$

and it is easy to check that this homomorphism is the inverse of the first.

An identical argument shows that $\Gamma(\tilde{\mathcal{E}}_d)$ is isomorphic to $E(\pi^{u'}, \pi^{v'})$, where $u' \equiv u \pmod d$ and $v' \equiv v \pmod d$. This shows the existence of the desired presentation for $st_*\tilde{\mathcal{E}}_d$.

It remains to show that the maps $st_*\tilde{c}_{d,d'}$ are power maps (2). Again, since the arguments are essentially identical for each pair d and d', it suffices to prove this in the case of $\tilde{c}_{r,\sigma}$ for some σ dividing r.

As above, we have $u + v = r$. Let σ be a divisor of r, and $d = r/\sigma$. Let u' be the smallest non-negative integer congruent to ud modulo r and v' be the smallest non-negative integer congruent to vd modulo r. Define integers u'' and v'' as

$$u'' = \frac{du - u'}{r} \text{ and } v'' = \frac{dv - v'}{r}.$$

The module $\Gamma(\tilde{\mathcal{E}}_r) \cong E(\pi^u, \pi^v)$ is generated by η_1, and η_2 with $\eta_1 = (\zeta_2, e^v\xi_1)$ and $\eta_2 = (h^u\zeta_1, \xi_2)$. Further, $\tilde{\mathcal{E}}_\sigma$ may be defined on U by $\langle \phi_1, \phi_2 | z\phi_2 = e^{u'}\phi_1,\ x\phi_1 = e^{v'}\phi_2\rangle$ and on V by $\langle \psi_1, \psi_2 | s\psi_2 = h^{v'}\psi_1,\ y\psi_1 = e^{u'}\psi_2\rangle$, so we may describe $st_*\tilde{\mathcal{E}}_\sigma$ as above: the module $\Gamma(\tilde{\mathcal{E}}_\sigma)$ is isomorphic to $E(\pi^{u'}, \pi^{v'})$, and is generated by $\gamma_1 = (\phi_2, e^{v'}\psi_1)$ and $\gamma_2 = (h^{u'}\phi_1, \psi_2)$.

We must show that $\eta_1^{d-i}\eta_2^i$ maps, via $st_*(\tilde{c}_{r,\sigma})$, to $\pi^{ui}x^{v''-i}\gamma_1$ for $0 \le i \le u''$ and to $\pi^{v(d-i)}y^{u''-(d-i)}\gamma_2$ for $u'' \le i \le d$.

We will do the first case—the second case is similar. The element $\eta_1^{d-i}\eta_2^i$ is of the form

$$\eta_1^{d-i}\eta_2^i = (\zeta_2, e^v\xi_1)^{d-i}(h^u\zeta_1, \xi_2)^i = (h^{ui}\zeta_1^i\zeta_2^{d-i}, e^{v(d-i)}\xi_1^{d-i}\xi_2^i),$$

so on U, this element $\eta_1^{d-i}\eta_2^i$ maps as

$$h^{ui}\zeta_1^i\zeta_2^{d-i} \mapsto x^{v''-i}e^{ui}h^{ui}\phi_2 = \pi^{ui}x^{v''-i}\phi_2.$$

On V, the element $\eta_1^{d-i}\eta_2^i$ maps as

$$e^{(d-i)v}\xi_1^{d-i}\xi_2^i \mapsto s^{v''-i}h^{iu}e^{(d-i)v}\psi_1.$$

It is straightforward to check that these are the same on $U \cap V$. But this is exactly the canonical d-th power map (2) for $E(\pi^v, \pi^u)^{\otimes d} \to E(\pi^{v'}, \pi^{u'})$, as desired. □

Remarks 2.4.2. (1) It is important to note that if any of the m_i is greater than $r-1$, Lemma 2.4.1 is no longer true. In particular, the sheaf $R^1st_*\tilde{\mathcal{E}}_r$ no longer vanishes in case 2 of the proof, and the subsequent fiber-to-family transitions are not valid.

(2) As was mentioned in the Introduction, the entire theory including the above proof can be reformulated in the language of twisted curves (orbicurves). The proof in the orbicurve formulation requires the use of the Abramovich-Vistoli stabilization of twisted stable maps [2, Prop 9.1.1] instead of the usual Behrend-Manin stabilization that we use here, but the cases and conditions that need to be checked are essentially the same in both approaches. For this paper we chose the torsion-free sheaf formulation because it is more concrete, closer to the physical origin of the theory, and consistent with the papers [27, 26, 24] where the virtual class on $\overline{\mathcal{M}}^{1/r}_{g,n}$ is constructed. It is also better suited to the treatment of the generalized descent property of Subsection 4.6, which provides an important link between ordinary (non-spin) Gromov-Witten invariants with descendants and spin Gromov-Witten invariants without descendants.

3. Cohomology classes

Here we introduce and study various cohomology classes in $H^\bullet(\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta),\mathbb{Q})$ necessary for constructing spin Gromov-Witten invariants and the corresponding CohFT.

3.1. **Tautological classes.**

There are many natural cohomology classes in $H^\bullet(\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta),\mathbb{Q})$. Of special interest are the tautological classes induced by the universal sections

$$\mathfrak{p}_i : \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta) \to \mathcal{C}^{1/r,\mathbf{m}}_{g,n}$$

corresponding to the marked points of the universal curve $\pi : \mathcal{C}_{g,n}^{1/r,\mathbf{m}} \to \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}(V,\beta)$. These are classes

$$\psi_i := c_1(\mathfrak{p}_i^*(\omega_\pi)) \quad \text{and} \quad \tilde{\psi}_i := c_1(\mathfrak{p}_i^*(\mathcal{E}_r)) \tag{6}$$

(and also classes $\tilde{\psi}_i^{(d)}$ for each divisor d of r). In [16] it is proved that these classes are closely related:

$$r\tilde{\psi}_i = (m_i + 1)\psi_i. \tag{7}$$

Although they will not be used in this paper, it is worth noting that the boundary classes, which are also of interest, have a combinatorial structure that is nicely described in terms of decorated graphs in a straightforward generalization of the methods of [16].

3.2. **Spin virtual class.**

Recall from [16, §4.1] that an r-spin virtual class on the stack of stable, r-spin curves gives, among other things, a cohomology class

$$c_{g,n}^{1/r}(\mathbf{m}) \in H^{2D}(\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}, \mathbb{Q}) \tag{8}$$

for every stable g, n, and r, (i.e., for $2g-2+n>0$). Here, the dimension D is

$$D = \frac{1}{r}\big((r-2)(g-1) + \sum_{i=1}^{n} m_i\big). \tag{9}$$

The collection of classes $c_{g,n}^{1/r}(\mathbf{m})$ is required to satisfy the axioms of convexity, cutting edges, vanishing, and forgetting tails.

Currently two different constructions of a spin virtual class $c^{1/r}$ are known, an algebro-geometric [27] and an analytic [24].

We can use the choice of an r-spin virtual class for stable r-spin curves to produce a similar r-spin class for all stacks of stable spin maps in the stable range of (g,n).

Definition 3.2.1. Given an r-spin virtual class $\{c_{g,n}^{1/r}(\mathbf{m}) \in H^{2D}(\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}, \mathbb{Q})\}$ satisfying the axioms of [16, §4.1], then for each V, and for each stable pair (g,n), we define the r-spin virtual class on $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}(V,\beta)$ by

$$\tilde{c}_{g,n}^{1/r}(\mathbf{m}) = \tilde{st}^* c_{g,n}^{1/r}(\mathbf{m}) \in H^{2D}(\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}(V,\beta), \mathbb{Q}). \tag{10}$$

In the case that (g,n) is not a stable pair (i.e., $2g-2+n \leq 0$), we define the r-spin virtual class directly. We do this first in genus zero.

Definition 3.2.2. If $g=0$ and $n<3$ then we define $\tilde{c}_{0,n}^{1/r}(\mathbf{m})$ to be the top Chern class of the dual of the first cohomology of the r-th root bundle $\mathcal{E}_r$; namely,

$$\tilde{c}_{0,n}^{1/r}(\mathbf{m}) = c_D(-R^1\pi_*\mathcal{E}_r), \tag{11}$$

where $\mathcal{E}_r$ is the r-th root of the universal spin structure $(\{\mathcal{E}_d\}, \{c_{d,d'}\})$ on the universal curve $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{0,n}^{1/r,\mathbf{m}}(V,\beta)$.

In the case that $g=1$ and $n=0$, the moduli space $\overline{\mathcal{M}}_{1,0}^{1/r}(V,\beta)$ decomposes into the disjoint union of d substacks, where d is the number of positive divisors of r (including 1 and r); these components correspond to the fact that (on the smooth locus) r-spin structures are in one-to-one correspondence with r-torsion points of the Jacobian of the underlying curve. No deformation of the underlying curve can take a point of order i to a point of order j unless $i = j$, so the moduli space breaks up into disjoint substacks

$$\overline{\mathcal{M}}^{1/r}_{1,0}(V,\beta) = \coprod_{\substack{i \mid r \\ 1\le i\le r}} \overline{\mathcal{M}}^{1/r,(i)}_{1,0}(V,\beta).$$

We call i the *index* of the substack if the r-th root is a point of exact order i in the Jacobian of the underlying curve.

Definition 3.2.3. If $g = 1$ and $n = 0$, define the r-spin virtual class $\tilde{c}^{1/r}_{1,0}(V,\beta)$ to be the following 0-dimensional class

$$\tilde{c}^{1/r}_{1,0} = \begin{cases} -(r-1) & \text{if the index is 1} \\ 1 & \text{otherwise.} \end{cases} \tag{12}$$

Proposition 3.2.4. *If $g = 0$ and $n < 3$, and if no marking m_i in $\mathbf{m}$ is equal to $r-1$, the r-spin virtual class $\tilde{c}^{1/r}_{0,n}(\mathbf{m})$ has dimension zero; and thus we have*

$$\tilde{c}^{1/r}_{0,n}(\mathbf{m}) = \begin{cases} 0 & \textit{if any } m_i = r-1 \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. The degree of the sheaf $\mathcal{E}_r$ is an integer and is given by

$$\deg \mathcal{E}_r = (2g - 2 - \sum m_i)/r,$$

hence when $g = 0$ we have

$$\sum m_i \equiv -2 \pmod{r}.$$

The dimension D of $\tilde{c}^{1/r}_{0,n}$ is

$$D = ((2-r) + \sum m_i)/r.$$

If $n = 0$ we have $\sum m_i = 0$, which implies $r = 2$, and we immediately have $D = 0$. If $2 \ge n \ge 1$, then since $0 \le m_i \le r-2$, we have $0 \le \sum m_i \le 2(r-2)$; and hence $\sum m_i = r-2$ is the only solution to the congruence $\sum m_i \equiv -2 \pmod{r}$. Consequently,

$$D = (2 - r + \sum m_i)/r = 0.$$

If any of the m_i are equal to $r-1$, then the argument in the proof of Axiom 4 in [16, Theorem 4.1] shows that $\tilde{c}^{1/r}$ must be zero. □

Theorem 3.2.5. *If $g = 0$, then $\tilde{c}^{1/r}_{0,n}(\mathbf{m})$ is the top Chern class $c_D(-R^1\tilde{\pi}_*\tilde{\mathcal{E}}_r)$ of the bundle whose fiber is the dual of the first cohomology of the r-th root $\tilde{\mathcal{E}}_r$ on the universal curve $\tilde{\pi} : \tilde{\mathcal{C}} \to \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}(V,\beta)$.*

Proof. For $n < 3$, this is true by definition.

In the case that $n \ge 3$, since $g = 0$, the r-spin virtual class $c^{1/r}_{0,n} \in H^{2D}(\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}, \mathbb{Q})$ is the top Chern class $c_D(-R^1\pi_*\mathcal{E}_r)$ of the first cohomology of the r-th root $\mathcal{E}_r$ on the universal curve $\pi : \mathcal{C} \to \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}$, by the convexity axiom of [16, §4.1].

We have the following commutative diagram.

$$\begin{tikzcd}
\tilde{\mathcal{C}} \arrow[r, "\phi"] \arrow[dr, "\tilde{\pi}"'] & \mathcal{C} \times_{\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}} \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}(V,\beta) \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathcal{C} \arrow[d, "\pi"] \\
 & \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}(V,\beta) \arrow[r, "\tilde{st}"] & \overline{\mathcal{M}}^{1/r}_{0,n}
\end{tikzcd}$$

Here ϕ is the natural map induced by $\tilde{\pi}$ and stabilization of $\tilde{\mathcal{C}}$. If $\tilde{\mathcal{E}}_r$ is the r-th root on $\tilde{\mathcal{C}}$, then by Lemma 2.4.1 and the universality of the sheaves involved, $\tilde{\phi}_*\tilde{\mathcal{E}}_r$ is isomorphic to the pullback $p_1^*\mathcal{E}_r$ of the r-th root $\mathcal{E}_r$ from $\mathcal{C}$, and $R^1\phi_*\tilde{\mathcal{E}}_r = 0$. By the Leray spectral sequence we have

$$R^1\tilde{\pi}_*\tilde{\mathcal{E}}_r = R^1 p_{2*}(p_1^*\mathcal{E}_r).$$

Even though the morphism $\tilde{st}$ is not flat, the natural map

$$\tilde{st}^* R^1\pi_*\mathcal{E}_r \longrightarrow R^1 p_{2*}(p_1^*\mathcal{E}_r)) \tag{13}$$

is an isomorphism. Indeed, since the morphism p_2 has relative dimension 1, for any sheaf F we have $R^2 p_{2*}F = 0$. This implies that the functor $R^1 p_{2*}\circ p_1^*$ is right exact and therefore by [9, III.12.5] the map (13) is an isomorphism.

Thus we have

$$c_D(-R^1 p_{2*}(p_1^*\mathcal{E}_r)) = \tilde{st}^* c_D(-R^1\pi_*\mathcal{E}_r) = \tilde{st}^* c^{1/r}_{0,n} = \tilde{c}^{1/r}_{0,n}(\mathbf{m}).$$

□

Remark 3.2.6. The proof of Theorem 3.2.5 depends upon the fact that the markings m_i in $\mathbf{m}$ lie in the range $0 \le m_i \le r-1$. In particular, when an m_i lies outside that range, Lemma 2.4.1 fails.

We shall also see (in Remark 4.6.2) that Theorem 3.2.5 is false in the case that any m_i is larger than $r-1$.

Definition 3.2.7. We define $[\overline{\mathcal{M}}^{1/r}_{g,n}(V,\beta)]^{virt}$ to be the pullback

$$[\overline{\mathcal{M}}^{1/r}_{g,n}(V,\beta)]^{virt} := \tilde{p}^*[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{virt}$$

of the usual virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{virt}$ of $\overline{\mathcal{M}}_{g,n}(V,\beta)$ via

$$\tilde{p} : \overline{\mathcal{M}}^{1/r}_{g,n}(V,\beta) \to \overline{\mathcal{M}}_{g,n}(V,\beta).$$

3.3. **Decomposition of classes.**

Using the notation of the commutative diagram (5), since $\tilde{ev}_i = ev_i \circ \tilde{p}$, for any $\gamma_1, \dots, \gamma_n \in H^\bullet(V, \mathbb{Q})$ we have the equality

$$\tilde{ev}_1^*\gamma_1 \cup \tilde{ev}_2^*\gamma_2 \cup \cdots \cup \tilde{ev}_n^*\gamma_n = \tilde{p}^*(ev_1^*\gamma_1 \cup \cdots \cup ev_n^*\gamma_n).$$

We also have the following important relation on pushforwards of classes, which is the crucial step in proving that the CohFT defined by stable r-spin maps is the tensor product of the CohFTs of r-spin curves and stable maps (Theorem 4.3.2).

Theorem 3.3.1 (Decomposition)**.** *Given any set $\{\gamma_1, \dots, \gamma_n\}$ of classes in $A^*(V)$ (or $H^\bullet(V)$), and given the r-spin virtual class $\tilde{c}^{1/r}$ on $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)$ defined by equations (10), (11) and (12), the relation*

(14)
$$q_*(\tilde{c}^{1/r} \cup \prod_{i=1}^n \tilde{ev}_i^*(\gamma_i) \cap [\overline{\mathcal{M}}^{1/r}_{g,n}(V,\beta)]^{\mathrm{virt}}) = p_* c^{1/r} \cup st_*(\prod_{i=1}^n ev_i^*(\gamma) \cap [\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}})$$

holds.

Proof. We will give the proof on the level of (operational) Chow groups A^* with notation as in [22, V §8]. From [22, VI §2] it will follow then that such results also hold for $H^\bullet(V)$.

To begin, let us fix some notation. We denote the identity maps on $\overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}^{1/r}_{g,n}$, $\overline{\mathcal{M}}_{g,n}(V,\beta)$, $\overline{\mathcal{M}}^{1/r}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n}(V,\beta)$, and $\overline{\mathcal{M}}^{1/r}_{g,n}(V,\beta)$ by $\mathbb{I}$, $\mathbb{I}_r$, $\mathbb{I}_V$, $\mathbb{I}_\times$, and $\mathbb{I}_{r,V}$, respectively. We have $c^{1/r} \in A^*(\overline{\mathcal{M}}^{1/r}_{g,n}) := \bar{A}^*(\mathbb{I}_r : \overline{\mathcal{M}}^{1/r}_{g,n} \to \overline{\mathcal{M}}^{1/r}_{g,n})$, and $\tilde{c}^{1/r} = \tilde{st}^*(c^{1/r}) \in A^*(\overline{\mathcal{M}}^{1/r}_{g,n}(V,\beta))$. We take γ_i in $A^*(V)$, so that $\tilde{ev}_i^*(\gamma_i)$ is in $A^*(\overline{\mathcal{M}}^{1/r}_{g,n}(V,\beta))$. Also, we have $[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}} \in A_*(\overline{\mathcal{M}}_{g,n}(V,\beta))$.

Finally, by
$$\tilde{c}^{1/r} \cup \prod_{i=1}^n \tilde{ev}_i^*(\gamma_i) \cap [\overline{\mathcal{M}}^{1/r}_{g,n}(V,\beta)]^{\mathrm{virt}}$$
we mean
$$\left(\tilde{c}^{1/r} \cup \prod_{i=1}^n \tilde{ev}_i(\gamma_i)\right)_{\mathbb{I}_{r,V}} \cap [\overline{\mathcal{M}}^{1/r}_{g,n}(V,\beta)]^{\mathrm{virt}}.$$

As in [22, V §8.9], for any morphism $Y \to X$, we define $f^* : A^*(X) \to A^*(Y)$ to be

$$f^*(\delta)_h \cap y := \delta_{f\circ h} \cap y, \tag{15}$$

where $\delta \in A^*(X)$ and $h : L \to Y$ is an arbitrary morphism, and $y \in A_*(L)$. We also define, for any proper, flat morphism $f : Y \to X$ of Deligne-Mumford stacks X and Y, the proper flat pushforward $f_\bullet : A^*(Y) \to A^*(X)$ to be

$$f_\bullet \alpha_g \cap c := f_*(\alpha_{f'_Y} \cap f^*(c)), \tag{16}$$

where $g : L \to X$ is an arbitrary morphism, α is an element of $A^*(Y)$, and c is an element of $A_*(X)$.

Remark 3.3.2. Note that part (ii) of Manin's definition in [22, V §8.9] of the operational Chow ring $A^*(M)$ for the identity morphism $\mathbb{I} : M \to M$ states that elements of $A^*(M)$ only need to commute with pullback along *representable*, flat morphisms of DM-stacks, despite the fact that standard definitions of general operational Chow rings require that these elements commute with pullback along *all* flat morphisms of DM-stacks (see Vistoli [28, 5.1.i] and Manin [22, V.8.1.i]).

In what we do below, we will need the definition of $A^*(M)$ that requires commutativity with all flat pullbacks; that is, we require the following.

Let $f : X \to Y$ be a flat morphism of Deligne-Mumford stacks, which is *not necessarily representable*, and let $h : Y \to Z$ be an arbitrary morphism of Deligne-Mumford stacks. For any $\sigma \in A^*(Z)$ and $y \in A_*(Y)$, we have

$$\sigma_{h\circ f} \cap f^*(y) = f^*(\sigma_h \cap y). \tag{17}$$

This seemingly minor difference in the definition of A^* allows us to prove a projection formula for non-representable morphisms.

Lemma 3.3.3. *Let $f : X \to Y$ be a proper, flat morphism of Deligne-Mumford stacks (which is not necessarily representable).*

1. *For an arbitrary morphism $h : L \to Y$ of Deligne-Mumford stacks we have*

$$h^* f_\bullet = f_{L\bullet} h_X^*. \tag{18}$$

2. ***(Projection formula for $f_\bullet$)*** *For any $\sigma \in A^*(X)$ and $\beta \in A^*(Y)$ we have*

$$f_\bullet(\sigma f^*(\beta)) = f_\bullet(\sigma)\beta. \tag{19}$$

Proof. For part 1 of the lemma, the same proof as given by Manin for this equation [22, V.8.30] works exactly for our case, too; nowhere is the representability of f used in Manin's proof.

For part 2, again Manin's proof of the projection formula [22, V.8.29] works for non-representable morphisms, the only change needed is that [22, V.8.22] (commutativity with flat, representable pullbacks) must be replaced by our equation (17) for non-representable, flat pullbacks. □

One more fact we will need in the proof of Theorem 3.3.1 is the commutativity with proper pushforwards required by the definition of A^* (cf. [22, V.8.21]); namely, if $p : P \to L$ is proper, and $h : L \to M$ is an arbitrary morphism, then by definition of $A^*(M)$, for any $\sigma \in A^*(M)$ and for any $y \in A_*(P)$ we have

$$\sigma_h \cap p_*(y) = p_*(\sigma_{hp} \cap y). \tag{20}$$

Now we may proceed with the proof of Theorem 3.3.1. We will refer throughout the proof to the notation of the commutative diagram (5).

Since q_1 is a birational map, it is a splitting morphism (i.e., $q_{1\bullet} q_1^* = \mathbb{I}_\times$, as a map on A^*). Moreover, the morphism st is proper,p is flat [11, Theorem 2.2] and proper [11, Theorem 2.3], and q_1 is flat and proper by Proposition 2.2.3. We have the following relations:

$$q_* \left(\left(\tilde{c}^{1/r} \cup \prod_{i=1}^{n} \tilde{ev}_i(\gamma_i) \right) \cap \tilde{p}^* [\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}} \right)$$

$$
\begin{aligned}
&= q_{2*}q_{1*}\left(q_1^*\left(pr_1^*c^{1/r}\cup pr_2^*\prod_{i-1}^n ev_i^*(\gamma_i)\right)_{\mathbb{I}_{r,V}}\cap q_1^*pr_2^*[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\right)\\
\text{(dfn. of } q_{1\bullet}) &= q_{2*}\left(q_{1\bullet}q_1^*\left(pr_1^*c^{1/r}\cup pr_2^*\prod_{i-1}^n ev_i^*(\gamma_i)\right)_{\mathbb{I}_\times}\cap pr_2^*[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\right)\\
(q_1 \text{ is splitting}) &= q_{2*}\left(\left(pr_1^*c^{1/r}\cup pr_2^*\prod ev_i^*(\gamma_i)\right)_{\mathbb{I}_\times}\cap pr_2^*[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\right)\\
&= st_*pr_{2*}\left(\left(pr_1^*c^{1/r}\cup pr_2^*\prod ev_i^*(\gamma_i)\right)_{\mathbb{I}_\times}\cap pr_2^*[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\right)\\
\text{(dfn. of } pr_{2\bullet}) &= st_*\left(pr_{2\bullet}\left(pr_1^*c^{1/r}\cup pr_2^*\prod ev_i^*(\gamma_i)\right)_{\mathbb{I}_V}\cap [\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\right)\\
\text{(prj. fmla. for } pr_{2\bullet}) &= st_*\left(\left(pr_{2\bullet}(pr_1^*c^{1/r})\cup \prod ev_i^*(\gamma_i)\right)_{\mathbb{I}_V}\cap [\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\right)\\
\text{(Equation 18)} &= st_*\left(\left((st^*p_\bullet c^{1/r})\cup \prod ev_i^*(\gamma_i)\right)_{\mathbb{I}_V}\cap [\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\right)\\
&= st_*\left((st^*p_\bullet c^{1/r})_{\mathbb{I}_V}\cap\left((\prod ev_i^*(\gamma_i))_{\mathbb{I}_V}\cap [\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\right)\right)\\
\text{(dfn. of } st^*) &= st_*\left((p_\bullet c^{1/r})_{st}\cap\left((\prod ev_i^*(\gamma_i))_{\mathbb{I}_V}\cap [\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\right)\right)\\
\text{(Equation 20)} &= (p_\bullet c^{1/r})_{\mathbb{I}}\cap st_*\left((\prod ev_i^*(\gamma_i))_{\mathbb{I}_V}\cap [\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\right).
\end{aligned}
$$

This completes the proof of Theorem 3.3.1. □

4. Gromov-Witten invariants and tensor products of CohFTs

4.1. **Standard Gromov-Witten invariants.**

Let V be a smooth projective variety. The moduli space of stable r-spin maps $\overline{\mathcal{M}}^{1/r}_{g,n}(V)$ gives rise to a set of correlators satisfying axioms analogous to those satisfied by Gromov-Witten invariants. This will follow from Theorem 4.3.2, which states that the CohFT associated to $\overline{\mathcal{M}}^{1/r}_{g,n}(V)$ is the tensor product of the Gromov-Witten CohFT with the r-spin CohFT.

We recall that the Gromov-Witten invariants $\Lambda^{(V)}_{g,n,\beta}: H^\bullet(V,\mathbb{C})\to H^\bullet(\overline{\mathcal{M}}_{g,n},\mathbb{C})$, defined as

$$\Lambda^{(V)}_{g,n,\beta}(\gamma_1,\dots,\gamma_n)=\mathrm{st}_*[(\prod_{i=1}^n ev_i^*\gamma_i)\cap[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}],$$

can be combined in formal power series as follows:

Definition 4.1.1. Let $\mathcal{R}$ denote the ring consisting of formal sums of expressions q^β with complex coefficients, where $\beta\in H_2(V,\mathbb{Z})$ belongs to the semigroup $B(V)$ of numerical equivalence classes such that $\beta\cdot L\geq 0$ for all ample divisor classes L in V. Furthermore, we impose on $\mathcal{R}$ the relations $q^{\beta_1+\beta_2}=q^{\beta_1}q^{\beta_2}$. We define $\Lambda^{(V)}_{g,n}: H^\bullet(V)^{\otimes n}\to H^\bullet(\overline{\mathcal{M}}_{g,n},\mathcal{R})$ as

$$\Lambda^{(V)}_{g,n}:=\sum_\beta q^\beta\Lambda^{(V)}_{g,n,\beta}.$$

Let $\Lambda^{(V)}$ denote the collection $\{\Lambda_{g,n}^{(V)}\}$ and 1 denote the unit in $H^\bullet(V)$.

Let η be the Poincaré pairing on $H^\bullet(V)$ and let $\eta_{\mu\nu} := \eta(e_\mu, e_\nu)$ be the coefficients of its matrix with respect to a basis $\{e_\mu\}$ for $H^\bullet(V)$. Denote by $(\eta^{\mu\nu})$ the inverse matrix of $(\eta_{\mu\nu})$.

Recall that a fundamental property of the Gromov-Witten invariants $\Lambda^{(V)}$ is that they define a CohFT on $(H^\bullet(V), \eta)$ with flat identity over $\mathcal{R}$ [19]. We refer the reader to [19, 16] for further details about CohFTs.

4.2. **Spin CohFT.**

Like ordinary Gromov-Witten invariants defined by means of stable maps, the spin Gromov-Witten invariants also form CohFTs.

Definition 4.2.1. Let $r \geq 2$ be an integer and let $(\mathcal{H}^{(r)}, \eta^{(r)})$ be the $(r-1)$-dimensional $\mathbb{C}$ vector space with basis $\{e_0, \ldots, e_{r-2}\}$ together with a metric

$$\eta^{(r)}{}_{m_1,m_2} := \eta^{(r)}(e_{m_1}, e_{m_2}) = \delta_{m_1+m_2,r-2}.$$

Let $c^{1/r}$ be an r-spin virtual class on $\overline{\mathcal{M}}_{g,n}^{1/r}$ satisfying the axioms from [16, §4.1]. Let

$$\Lambda_{g,n}^{(r)} : \mathcal{H}^{(r)\otimes n} \to H^\bullet(\overline{\mathcal{M}}_{g,n})$$

be defined by

$$\Lambda_{g,n}^{(r)}(e_{m_1}, \ldots, e_{m_n}) := r^{1-g} p_* c_{g,n}^{1/r,\mathbf{m}} \tag{21}$$

for all nonnegative numbers g, n such that $2g-2+n > 0$ where $p : \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}} \to \overline{\mathcal{M}}_{g,n}$. Finally, let $\Lambda^{(r)}$ denote the collection $\{\Lambda_{g,n}^{(r)}\}$.

Remark 4.2.2. As in the case of the CohFT based on ordinary stable maps, the classes $\{\Lambda_{g,n}^{(r)}\}$ *a priori* may depend on the choice of the spin virtual class $c^{1/r}$. Currently, there exist two different constructions of a candidate for such class on $\overline{\mathcal{M}}_{g,n}^{1/r}$: an algebro-geometric construction of [27], resembling algebraic constructions of the virtual fundamental class, and an analytic construction of [24] developing Witten's original idea [29]. While it is not known yet whether these constructions give the same class for all g and r, they agree when $g = 0$ (and any r) or $r = 2$ (and any g). In these cases, any class satisfying the axioms must be equal to the class constructed in [16] and therefore the resulting classes $\{\Lambda_{g,n}^{(r)}\}$ and the corresponding correlators do not depend on this choice.

Theorem 4.2.3 ([16, Theorem 3.8])**.** *For each integer $r \geq 2$, the triple $(\mathcal{H}^{(r)}, \eta^{(r)}, \Lambda^{(r)})$ forms a CohFT with flat identity e_0. It is called the r-spin CohFT.*

Since the space $\overline{\mathcal{M}}_{g,n}^{1/r}$ is associated to the r-spin CohFT, and the space $\overline{\mathcal{M}}_{g,n}(V)$ is associated to Gromov-Witten theory, it is natural to ask if there is a natural CohFT associated to the space $\overline{\mathcal{M}}_{g,n}^{1/r}(V, \beta)$. The answer is yes, and this CohFT is the tensor product of the other two.

4.3. **Tensor products of CohFTs.**

The category of cohomological field theories has a canonical tensor product operation (see [21]). This reflects the fact that the diagonal map $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g,n}$ is a coproduct with respect to the composition maps of the modular operad $\{H_\bullet(\overline{\mathcal{M}}_{g,n})\}$.

In the case of Gromov-Witten invariants, Behrend [4] proved that the CohFT arising from $\overline{\mathcal{M}}_{g,n}(V' \times V'')$ is the tensor product of that arising from $\overline{\mathcal{M}}_{g,n}(V')$ and $\overline{\mathcal{M}}_{g,n}(V'')$. Restricting to genus zero, one can regard this as a deformation of the Künneth theorem. Similarly, it was shown in [17] that the tensor product of an r-spin CohFT and an r'-spin CohFT can be geometrically realized by means of the moduli space of (r, r')-spin curves. To complete this picture, we need to provide an intersection-theoretical description of the tensor product of the Gromov-Witten theory with the r-spin CohFT.

Definition 4.3.1. Let $(H^\bullet(V,\mathbb{C}), \eta_P)$ denote the cohomology of V together with its Poincaré pairing η_P. Let $(\mathcal{H}^{(V,r)}, \eta)$ denote the tensor product of $(H^\bullet(V), \eta_P)$ with $(\mathcal{H}^{(r)}, \eta^{(r)})$. For each stable pair (g, n) and $\beta \in H_2(V, \mathbb{Z})$, define the *(cohomological) correlators* (or the spin Gromov-Witten invariants) to be linear maps

$$\Lambda^{(V,r)}_{g,n,\beta} : \mathcal{H}^{(V,r)} \to H^\bullet(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$$

given by

$$\Lambda^{(V,r)}_{g,n,\beta}(\gamma_1 \otimes e_{m_1}, \ldots, \gamma_n \otimes e_{m_n}) = Q_*[(\tilde{c}^{1/r,\mathbf{m}}_{g,n} \prod_{i=1}^{n} ev_i^* \gamma_i) \cap [\overline{\mathcal{M}}^{1/r}_{g,n}(V, \beta)]^{\mathrm{virt}}], \tag{22}$$

where $Q : \overline{\mathcal{M}}^{1/r}_{g,n}(V) \to \overline{\mathcal{M}}_{g,n}$ is the morphism that forgets both the stable map and the r-spin structure, $[\overline{\mathcal{M}}^{1/r}_{g,n}(V, \beta)]^{\mathrm{virt}}$ is the virtual fundamental class of $\overline{\mathcal{M}}^{1/r}_{g,n}(V)$, and $\gamma_i \otimes e_{m_i} \in \mathcal{H}^{(V,r)}$.

The following theorem holds.

Theorem 4.3.2. *Let* $\Lambda^{(V,r)}_{g,n} : \mathcal{H}^{(V,r)\otimes n} \to H^\bullet(\overline{\mathcal{M}}_{g,n}, \mathcal{R})$, *where*

$$\Lambda^{(V,r)}_{g,n} := \sum_\beta q^\beta \Lambda^{(V,r)}_{g,n,\beta}.$$

Let $\Lambda^{(V,r)}$ *denote the collection* $\{\Lambda^{(V,r)}_{g,n}\}$. *The collection* $(H^\bullet(V,\mathcal{R}), \eta, \Lambda)$ *forms a CohFT (over the ground ring* $\mathcal{R}$*) with flat identity* $1 \otimes e_0$ *and is the tensor product of the CohFTs* $(H^\bullet(V,\mathcal{R}), \eta, \Lambda^{(V)})$ *and* $(\mathcal{H}^{(r)}, \eta^{(r)}, \Lambda^{(r)})$.

Proof. This is an immediate consequence of Theorem 3.3.1. □

The r-spin CohFTs behave as though the elements of $\mathcal{H}^{(r)}$ were cohomology classes of fractional dimension, similar to the orbifold cohomology classes of Chen and Ruan [6]. Since r-spin CohFTs correspond to the case of r-spin maps into a point, the elements of $B(V)$ in that theory are all trivial. However, the theory associated to r-spin maps into a general target V does satisfy axioms analogous to those of Gromov-Witten theory. In particular, this theory, like the Gromov-Witten theory, is of qc-type in the sense of [22, 23].

4.4. **Spin Gromov-Witten invariants.**

The classes $\Lambda^{(V,r)}_{g,n,\beta}$ have properties analogous to those of Gromov-Witten invariants.

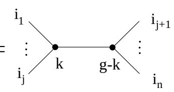
Theorem 4.4.1. *Let* (g, n) *be a stable pair of integers. The collection* $\{\Lambda^{(V,r)}_{g,n,\beta}\}$ *satisfies the following properties:*

1. *(Effectivity)* $\Lambda^{(V,r)}_{g,n,\beta} = 0$ *if* $\beta \notin B(V)$.

2. *(S_n-Equivariance) Each map $\Lambda^{(V,r)}_{g,n,\beta}$ is S_n-equivariant.*
3. *(Degeneration Axioms) Given a basis $\{\mathbf{e}_\mu\}$ for $\mathcal{H}^{(V,r)}$, let ${\eta^{(V,r)}}_{\mu\nu} := \eta^{(V,r)}(\mathbf{e}_\mu, \mathbf{e}_\nu)$ and $({\eta^{(V,r)}}^{\mu\nu})$ denote the inverse matrix.*
   (a) *Let*

$$\rho_{\Gamma_{\mathrm{tree}}} : \overline{\mathcal{M}}_{k,j+1} \times \overline{\mathcal{M}}_{g-k,n-j+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

be the gluing map corresponding to the stable graph

$\Gamma_{\mathrm{tree}} =$  .

The forms $\Lambda^{(V,r)}_{g,n\beta}$ satisfy the composition property:

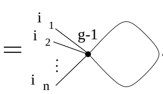
$$\rho^*_{\Gamma_{\mathrm{tree}}}\Lambda^{(V,r)}_{g,n,\beta}(\gamma_1,\gamma_2,\ldots,\gamma_n) = \sum_{\beta_1+\beta_2=\beta} \Lambda^{(V,r)}_{k,j+1,\beta_1}(\gamma_{i_1},\ldots,\gamma_{i_j},\mathbf{e}_\mu){\eta^{(V,r)}}^{\mu\nu} \otimes \Lambda^{(V,r)}_{g-k,n-j+1,\beta_2}(\mathbf{e}_\nu,\gamma_{i_{j+1}},\ldots,\gamma_{i_n})$$

for all $\gamma_i \in \mathcal{H}^{(V,r)}$.

   (b) *Let*

$$\rho_{\Gamma_{\mathrm{loop}}} : \overline{\mathcal{M}}_{g-1,n+2} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

be the gluing map corresponding to the stable graph

$\Gamma_{\mathrm{loop}} =$  .

The forms $\Lambda^{(V,r)}_{g,n\beta}$ satisfy the composition property:

$$\rho^*_{\Gamma_{\mathrm{loop}}}\Lambda^{(V,r)}_{g,n,\beta}(\gamma_1,\gamma_2,\ldots,\gamma_n) = \Lambda^{(V,r)}_{g-1,n+2,\beta}(\gamma_1,\gamma_2,\ldots,\gamma_n,\mathbf{e}_\mu,\mathbf{e}_\nu){\eta^{(V,r)}}^{\mu\nu}$$

for all $\gamma_i \in \mathcal{H}^{(V,r)}$.

4. *(Identity Axiom) Let $\mathbf{1} := 1 \otimes e_0$, where 1 is the unit in $H^\bullet(V)$ and e_0 the unit of $\mathcal{H}^{(r)}$. We have*

$$\Lambda^{(V,r)}_{g,n+1,\beta}(\gamma_1,\ldots,\gamma_n,\mathbf{1}) = \pi^*\Lambda^{(V,r)}_{g,n,\beta}(\gamma_1,\ldots,\gamma_n)$$

for all $\gamma_i \in \mathcal{H}^{(V,r)}$, where $\pi : \overline{\mathcal{M}}_{g,n+1}(V) \to \overline{\mathcal{M}}_{g,n}$ is the forgetful morphism.

5. *(Dimension Axiom) Let K_V denote the canonical class on V. The map $\Lambda^{(V,r)}_{g,n,\beta}$ of $\mathbb{Z}$-graded modules must be homogeneous of degree*

$$\left|\Lambda^{(V,r)}_{g,n,\beta}\right| = 2\int_\beta K_V + 2(g-2)\dim_{\mathbb{C}} V + \frac{2}{r}(r-2)(g-1).$$

6. *(Divisor Axiom) Let $\alpha \otimes e_0$ belong to $H^2(V) \otimes \mathcal{H}^{(r)}$. We have*

$$\pi_*\Lambda^{(V,r)}_{g,n+1,\beta}(\gamma_1,\ldots,\gamma_n,\alpha\otimes e_0) = \Lambda^{(V,r)}_{g,n,\beta}(\gamma_1,\ldots,\gamma_n)\int_\beta \alpha,$$

for all $\gamma_i \in \mathcal{H}^{(V,r)}$, where $\pi : \overline{\mathcal{M}}_{g,n+1}(V) \to \overline{\mathcal{M}}_{g,n}$ is the forgetful morphism.

7. *(Mapping to a Point Axiom)*

$$\Lambda^{(V,r)}_{g,n}(\gamma_1 \otimes e_{m_1},\ldots,\gamma_n \otimes e_{m_n}) = p_{2*}\left[p_1^*(\prod_{i=1}^n \gamma_i) \cup c_d(TV \boxtimes L)\right] \cup p_* c_{g,n}^{1/r,\mathbf{m}}$$

for all $\gamma_i \in H^\bullet(V)$, where $p_1 : V \times \overline{\mathcal{M}}_{g,n} \to V$ and $p_2 : V \times \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ are the canonical projections, TV is the tangent bundle, $L = R^1\pi_*\mathcal{O}_{\mathcal{C}_{g,n}}$ where $\mathcal{O}_{\mathcal{C}_{g,n}}$ is the structure sheaf on the universal curve $\pi : \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, and $d = g \dim_{\mathbb{C}} V$ (the rank of $TV \otimes L$). Finally, $p : \overline{\mathcal{M}}^{1/r}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is the morphism forgetting the spin structure and $\mathbf{m} = (m_1, \dots, m_n)$.

Proof. All axioms follow immediately from Theorem 3.3.1 and the corresponding properties of usual Gromov-Witten invariants [19]. □

4.5. **Potential functions.**

Recall the potential functions associated to $\overline{\mathcal{M}}_{g,n}(V)$.

Definition 4.5.1. Consider the *correlation functions*

$$\langle \tau_{a_1}(\gamma_1) \dots \tau_{a_n}(\gamma_n) \rangle_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}} \prod_{i=1}^{n} (\psi_i^{a_i} ev_i^* \gamma_i)$$

for all integers $a_1, \dots, a_n \geq 0$ and $\gamma_1, \dots, \gamma_n$ in $H^\bullet(V)$. Correlation functions such that some of the a_i are nonzero are called *gravitational descendants*.

The *large phase space potential (function)* associated to $\overline{\mathcal{M}}_{g,n}(V)$ is

$$\Phi^{(V)}(\mathbf{t}) := \sum_{g \geq 0} \lambda^{2g-2} \Phi_g^{(V)}(\mathbf{t}) \in \lambda^{-2}\mathcal{R}[[\lambda^2]][[t_a^\alpha]],$$

where

$$\Phi_g^{(V)}(\mathbf{t}) := \sum_{\beta \in B(V)} \langle \exp(\mathbf{t} \cdot \boldsymbol{\tau}) \rangle_{g,\beta} q^\beta$$

and

$$\mathbf{t} \cdot \boldsymbol{\tau} := \sum_{a \geq 0} \sum_{\alpha} t_a^\alpha \tau_a(\varepsilon_\alpha),$$

relative to a basis $\{\varepsilon_\alpha\}$ for $H^\bullet(V)$ such that ε_0 is the identity.

The *small phase space potential (function)*, $\Phi^{(V)}(\mathbf{x})$ where $\mathbf{x} = (x^1, \dots, x^n)$ are coordinates on $H^\bullet(V)$ relative to the basis $\{\varepsilon_\alpha\}$, is obtained from $\Phi^{(V)}(\mathbf{t})$ by setting $x^\alpha := t_0^\alpha$ and $t_a^\alpha := 0$ for all $a \geq 1$ and all α.

There are analogous potential functions associated to $\overline{\mathcal{M}}^{1/r}_{g,n}(V)$.

Definition 4.5.2. Consider the *correlation functions*

$$\langle \tau_{a_1}(\gamma_1 \otimes e_{m_1}) \dots \tau_{a_n}(\gamma_n \otimes e_{m_n}) \rangle_{g,\beta} := \int_{[\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)]^{\mathrm{virt}}} r^{1-g} \tilde{c}^{1/r}(\mathbf{m}) \prod_{i=1}^{n} (\psi_i^{a_i} ev_i^* \gamma_i)$$

for integers $a_1, \dots, a_n \geq 0$, $\gamma_1, \dots, \gamma_n \in H^\bullet(V)$, and $e_{m_1}, \dots, e_{m_n} \in \mathcal{H}^{(r)}$. Correlation functions such that some of the a_i are nonzero are called *gravitational descendants*.

The *large phase space potential (function)* associated to $\overline{\mathcal{M}}^{1/r}_{g,n}(V)$ is

$$\Phi^{(V,r)}(\mathbf{u}) := \sum_{g \geq 0} \lambda^{2g-2} \Phi_g^{(V,r)}(\mathbf{u}) \in \lambda^{-2}\mathcal{R}[[\lambda^2]][[\mathcal{H}^{(V,r)}]],$$

where

$$\Phi_g^{(V,r)}(\mathbf{u}) := \sum_{\beta \in B(V)} \langle \exp(\mathbf{u} \cdot \boldsymbol{\tau}) \rangle_{g,\beta} q^\beta$$

and

$$\mathbf{u}\cdot\boldsymbol{\tau} := \sum_{a\geq 0}\sum_{\alpha,m} u_a^{\alpha,m}\tau_a(\varepsilon_\alpha\otimes e_m),$$

relative to the basis $\{\varepsilon_\alpha\otimes e_m\}$ for $\mathcal{H}^{(V,r)}$.

The *small phase space potential (function)*, $\Phi^{(V,r)}(\mathbf{y})$ where $\mathbf{y}$ consists of coordinates $\{y^{\alpha,m}\}$ on $H^\bullet(V)$ relative to the basis $\{\varepsilon_\alpha\otimes e_m\}$, is obtained from $\Phi^{(V,r)}(\mathbf{u})$ by setting $y^{\alpha,m} := u_0^{\alpha,m}$ and $u_a^{\alpha,m} := 0$ for all $a\geq 1$ and all α, m.

Theorem 4.5.3. *The small phase space potential function $\Phi^{(V,r)}(\mathbf{y})$ is completely determined by the potential $\Phi^{(V)}(\mathbf{x})$, the cohomological correlators $\{\Lambda_{g,n}^{(V)}\}$, and $\{\Lambda_{g,n}^{(r)}\}$.*

Proof. Theorem 3.3.1 shows that the intersection numbers $\langle\gamma_1\otimes e_{m_1}\cdots\gamma_n\otimes e_{m_n}\rangle_{g,n}$ are completely determined by the classes $\{\Lambda_{g,n}^{(V)}\}$ and $\{\Lambda_{g,n}^{(r)}\}$ if (g,n) is stable.

We must still address the unstable cases—when $(g,n)\in\{(0,0),\ (0,1),\ (0,2),\ (1,0)\}$. But by Proposition 3.2.4 and Definition 3.2.3, these are always of dimension zero.

Let $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}} := \coprod_i \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},(i)}$, where $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},(i)}$ are the connected components of $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$, and let $\tilde{p}_{(i)} : \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},(i)}(V,\beta)\to\overline{\mathcal{M}}_{g,n}(V,\beta)$ be the morphisms forgetting the r-spin structure. Furthermore, let $\tilde{c}^{1/r,\mathbf{m},(i)}$ be $\tilde{c}^{1/r}$ restricted to $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},(i)}$ and let us assume that $\tilde{c}^{1/r,\mathbf{m},(i)}$ is zero dimensional. For all $\boldsymbol{\gamma}\otimes\mathbf{e} := \gamma_1\otimes e_{m_1}\cdots\gamma_n\otimes e_{m_n}$ in $\mathcal{H}^{(V,r)}$, we have

$$\begin{aligned}
\langle\boldsymbol{\gamma}\otimes\mathbf{e}\rangle_{g,\beta} &= r^{1-g}\int(\tilde{\mathbf{ev}}^*\boldsymbol{\gamma}\cup\tilde{c}^{1/r})\cap[\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}(V,\beta)]^{\mathrm{virt}}\\
&= \sum_i \tilde{c}_{g,n}^{1/r,\mathbf{m},(i)} r^{1-g}\int\tilde{\mathbf{ev}}^*\boldsymbol{\gamma}\cap[\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}(V,\beta)]^{\mathrm{virt}}\\
&= \sum_i \tilde{c}_{g,n}^{1/r,\mathbf{m},(i)} r^{1-g}\int\tilde{p}_{(i)}^*\mathbf{ev}^*\boldsymbol{\gamma}\cap\tilde{p}_{(i)}^*[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\\
&= \sum_i \tilde{c}_{g,n}^{1/r,\mathbf{m},(i)} r^{1-g}\int(\mathbf{ev}^*\boldsymbol{\gamma})_{\tilde{p}_{(i)}}\cap\tilde{p}_{(i)}^*[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\\
&= \sum_i \tilde{c}_{g,n}^{1/r,\mathbf{m},(i)} r^{1-g}\int\tilde{p}_{(i)}^*(\mathbf{ev}^*\boldsymbol{\gamma}\cap[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}})\\
&= \sum_i \tilde{c}_{g,n}^{1/r,\mathbf{m},(i)} r^{1-g}\deg(\tilde{p}_{(i)})\int\mathbf{ev}^*\boldsymbol{\gamma}\cap[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}\\
&= \sum_i \tilde{c}_{g,n}^{1/r,\mathbf{m},(i)} r^{1-g}\deg(\tilde{p}_{(i)})\langle\boldsymbol{\gamma}\rangle_{g,\beta},
\end{aligned}$$

where deg denotes the (orbifold) degree of $\tilde{p}_{(i)}$. This completes the proof. □

4.6. **The descent property.**

In this subsection, we show that when $g=0$, our constructions on $\overline{\mathcal{M}}_{0,n}^{1/r}(V,\beta)$ satisfy a generalization of the so-called descent property (introduced in [15]). This property of r-spin invariants gives a geometric origin for the ψ classes (at least in genus zero) in the definition of the usual Gromov-Witten invariants of V.

It may seem curious that $\overline{\mathcal{M}}^{1/r}_{g,n}(V,\beta)$ is defined to be the disjoint union of $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}(V,\beta)$, where the n-tuple of nonnegative integers $\mathbf{m} = (m_1,\dots,m_n)$ is required to satisfy $m_i \leq r-1$ for all $i = 1,\dots,n$. The latter restriction, however, is reasonable because of the isomorphism

$$\overline{\mathcal{M}}^{1/r,\widetilde{\mathbf{m}}}_{g,n}(V,\beta) \longrightarrow \overline{\mathcal{M}}^{1/r,\widetilde{\mathbf{m}}+r\boldsymbol{\delta}_i}_{g,n}(V,\beta)$$

from Proposition 2.1.5, where $i = 1,\dots,n$, $\boldsymbol{\delta}_i$ is the n-tuple whose i-th component is 1 and the rest are zero, and $\widetilde{\mathbf{m}} := (\tilde{m}_1,\dots,\tilde{m}_n)$ is any n-tuple of nonnegative integers.

On the other hand, in genus zero the classes $c^{1/r}(\widetilde{\mathbf{m}})$ change under this identification in the following manner.

Theorem 4.6.1. *(The descent property) Let $\widetilde{\mathbf{m}} = (\tilde{m}_1,\dots,\tilde{m}_n)$ be an n-tuple of nonnegative integers and let $\mathbf{m} = (m_1,\dots,m_n)$ be the reduction of $\widetilde{\mathbf{m}} \pmod r$ (i.e., $\widetilde{\mathbf{m}} \equiv \mathbf{m} \pmod r$ and $0 \leq m_i \leq r-1$ for $i = 1,\dots,n$).*

Let $\tilde{c}^{1/r}(\widetilde{\mathbf{m}})$ be the top Chern class of the vector bundle $R^1\pi_*\mathcal{E}(\widetilde{\mathbf{m}})^*$ on $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}$.

The following equation is satisfied on $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}$ for all $i = 1,\dots,n$, where $\boldsymbol{\delta}_i$ is the n-tuple whose i-th component is 1 and the rest are zero:

$$r\tilde{c}^{1/r}(\widetilde{\mathbf{m}} + r\boldsymbol{\delta}_i) = -(\tilde{m}_i + 1)\psi_i\tilde{c}^{1/r}(\widetilde{\mathbf{m}}). \tag{23}$$

Proof. The proof is identical to the case of $\overline{\mathcal{M}}^{1/r}_{g,n}$ in [15]. It follows from the short exact sequence

$$0 \longrightarrow \mathcal{E}_r(\widetilde{\mathbf{m}} + r\boldsymbol{\delta}_i) \longrightarrow \mathcal{E}_r(\widetilde{\mathbf{m}}) \longrightarrow \sigma_i^*\mathcal{E}_r(\widetilde{\mathbf{m}}) \longrightarrow 0$$

and the fact that

$$r\tilde{\psi}_i := rc_1(\sigma_i^*\mathcal{E}_r(\widetilde{\mathbf{m}})) = (m_i + 1)\psi_i$$

for all $i = 1,\dots,n$, which follows from an immediate generalization of Proposition 2.2 from [16]. $\square$

Remark 4.6.2. The descent property holds on both $\overline{\mathcal{M}}^{1/r}_{0,n}$ and $\overline{\mathcal{M}}^{1/r}_{0,n}(V,\beta)$, but the ψ classes on $\overline{\mathcal{M}}^{1/r}_{0,n}(V,\beta)$ are not pullbacks of the corresponding ψ classes on $\overline{\mathcal{M}}^{1/r}_{g,n}$—just as in the case of stable maps, they differ by divisors that are collapsed under the stabilization map (see [20, 22]). This illustrates the fact, alluded to in Remarks 2.4.2 and 3.2.6, that when any $\tilde{m}_i$ is larger than $r-1$, the class $\tilde{\mathrm{st}}^* c^{1/r}(\widetilde{\mathbf{m}})$ is not equal to the class $\tilde{c}^{1/r}(\widetilde{\mathbf{m}})$.

The previous theorem motivates the following generalization of the small phase space potential function in genus zero.

Definition 4.6.3. Let the n-tuples $\widetilde{\mathbf{m}} = (\tilde{m}_1,\dots,\tilde{m}_n)$ and $\mathbf{m}$ and the class $\tilde{c}^{1/r}(\widetilde{\mathbf{m}})$ on $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}$ be the same as in the previous theorem.

Define the *correlation functions*

$$\langle\tilde{\tau}_0(\gamma_1 \otimes e_{\tilde{m}_1})\dots\tilde{\tau}_0(\gamma_n \otimes e_{\tilde{m}_n})\rangle_{0,\beta} := \int_{[\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}(V,\beta)]^{\mathrm{virt}}} r\tilde{c}^{1/r}(\widetilde{\mathbf{m}})\prod_{i=1}^{n}\tilde{ev}_i^*\gamma_i.$$

Consider the analog of the genus zero small phase space potential

$$\widetilde{\Phi}_0^{(V,r)}(\tilde{\boldsymbol{t}}) \in \mathcal{R}[[\lambda^2]][[\tilde{t}^{\alpha,\tilde{m}}]],$$

where

$$\widetilde{\Phi}_0^{(V,r)}(\tilde{\boldsymbol{t}}) := \sum_{\beta \in B(V)} \langle \exp(\tilde{\boldsymbol{t}} \cdot \tilde{\boldsymbol{\tau}}) \rangle_{0,\beta} q^{\beta},$$

and

$$\tilde{\boldsymbol{t}} \cdot \tilde{\boldsymbol{\tau}} := \sum_{\alpha, \tilde{m}} \tilde{t}^{\alpha,\tilde{m}} \tilde{\tau}_0(\varepsilon_\alpha \otimes e_{\tilde{m}}),$$

where the last sum runs over all α and all nonnegative integers $\tilde{m}$.

Corollary 4.6.4. *Let $r \geq 2$ be an integer. The potential functions $\widetilde{\Phi}_0^{(V,r)}(\tilde{\boldsymbol{t}})$ and $\Phi_0^{(V,r)}(\mathbf{u})$ are equal after making the assignment:*

$$\tilde{t}^{\alpha,(ar+m)} := \frac{(-1)^a r^a}{[r(a-1)+m+1]_r} u_a^{\alpha,m},$$

where a and m are nonnegative integers such that $m \leq r-1$ and

$$[r(a-i)+m+1]_r := \prod_{i=1}^{a} (r(a-i)+m+1).$$

5. Examples and special cases

5.1. The case of $r = 2$.

In [16, 29], the virtual class $c^{1/r}(\mathbf{m})$ when $r = 2$ was constructed for all genera and n-tuples $\mathbf{m} = (m_1, \ldots, m_n)$ with $0 \leq m_i \leq 1$. It was shown that the $r = 2$ case reduced to the Gromov-Witten invariants of a point. A similar result is true for all 2-spin Gromov-Witten invariants.

Theorem 5.1.1. *For a pair of nonnegative integers (g,n) and $\beta \in H_2(V,\mathbb{Z})$ let $\tilde{p} : \overline{\mathcal{M}}_{g,n}^{1/2}(V,\beta) \to \overline{\mathcal{M}}_{g,n}(V,\beta)$ be the map forgetting the spin structure. For $i = 1, \ldots, n$, let $\gamma_i \otimes e_0$ belong to $\mathcal{H}^{(V,r)}$, then*

$$2^{1-g} \tilde{p}_* \left(\tilde{c}^{1/2}(\mathbf{0}) \prod_{i=1}^{n} (\tilde{ev}_i^* \gamma_i) \cap [\overline{\mathcal{M}}_{g,n}^{1/2}(V,\beta)]^{\mathrm{virt}} \right) = \prod_{i=1}^{n} (ev_i^* \gamma_i) \cap [\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}.$$

Consequently, the large phase space potential functions $\Phi^{(V,2)}(\mathbf{u})$ and $\Phi^V(\mathbf{t})$ agree after setting $u_a^{(\alpha,0)} = t_a^\alpha$.

Proof. This was proved in the case where V is a point in [16]. The same proof goes through here using the definition of $\tilde{c}^{1/r}$ (which is now defined in the unstable range) and the fact that $[\overline{\mathcal{M}}_{g,n}^{1/2}(V,\beta)]^{\mathrm{virt}} = \tilde{p}^* [\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}$. □

5.2. The case of $g = 0$ and $\beta = 0$.

Genus zero Gromov-Witten invariants of V give rise to the quantum cohomology of V, which is a certain deformation of the cup product on $H^\bullet(V)$. The cup product itself appears as the $\beta = 0$ part of the genus zero potential function. Similarly, the Frobenius structure associated to $\overline{\mathcal{M}}_{g,n}^{1/r}(V)$ can be regarded as a deformation of the following commutative, associative product on $\mathcal{H}^{(V,r)}$.

Proposition 5.2.1. *Let V be a smooth projective variety and $n \geq 3$ be an integer. Let $\gamma_1, \ldots, \gamma_n$ belong to $H^\bullet(V)$ and $e_0, \ldots, e_{r-2}$ be the standard basis in $\mathcal{H}^{(r)}$, then*

$$\langle \gamma_1 \otimes e_{m_1} \cdots \gamma_n \otimes e_{m_n} \rangle_{g,\beta=0} = \int_{\overline{\mathcal{M}}_{0,n}^{1/r,\mathbf{m}}} c^{1/r}(\mathbf{m}) \int_V \gamma_1 \cup \ldots \cup \gamma_n.$$

Proof. This follows from the *Mapping to a Point* property. □

5.3. **The case of $g = 0$, $r = 3$, and $V = \mathbb{P}^1$.** Throughout this section let $r = 3$ and $V = \mathbb{P}^1$. We will now compute its genus zero small phase potential function, denoted by

$$\chi(\mathbf{t}) := \Phi_0^{(\mathbb{P}^1,3)}(\mathbf{t}),$$

where $\mathbf{t}$ is a set of coordinates $t^{\alpha,m}$ associated to the basis $\{\tau_{\alpha,m} := \varepsilon_\alpha \otimes e_m\}$ (where $\alpha = 0,1$ and $m = 0,1$) for $\mathcal{H}^{(\mathbb{P}^1,3)}$. Here ε_0 is the identity element in $H^\bullet(\mathbb{P}^1)$ and ε_1 is the element in $H^2(\mathbb{P}^1)$ Poincaré dual to a point. The metric in this basis is

$$\eta_{(\alpha_1,m_1),(\alpha_2,m_2)} := \eta(\varepsilon_{\alpha_1} \otimes e_{m_1}, \varepsilon_{\alpha_2} \otimes e_{m_2}) = \delta_{\alpha_1+\alpha_2,1}\delta_{m_1+m_2,1}.$$

The potential function can be broken into two pieces:

$$\chi(\mathbf{t}) = \chi_{\beta=0}(\mathbf{t}) + \Psi(\mathbf{t}),$$

where $\chi_{\beta=0}(\mathbf{t})$ consists of only those terms corresponding to the moduli spaces $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 0)$; while $\Psi(\mathbf{t})$ contains the contributions ("instanton corrections") from $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, \beta)$ where $\beta \neq 0$. Corollary 5.2.1 implies that

$$\chi_{\beta=0}(\mathbf{t}) = \frac{1}{2}t^{1,1}(t^{0,0})^2 + t^{0,0}t^{0,1}t^{1,0} + \frac{1}{18}t^{1,1}(t^{0,1})^3. \tag{24}$$

Theorem 4.4.1 implies that

$$\Psi(\mathbf{t}) = \sum_{\beta\geq 1}\sum_{n_1,n_2\geq 0} q^\beta \frac{(t^{0,1})^{n_1}(t^{1,0})^{n_2}(t^{1,1})^{6\beta+2n_1-5}}{n_1!n_2!(6\beta+2n_1-5)!}\langle \tau_{0,1}^{n_1}\tau_{1,0}^{n_2}\tau_{1,1}^{6\beta+2n_1-5}\rangle_\beta. \tag{25}$$

Furthermore, Theorem 4.4.1 implies that the potential function must satisfy the WDVV equation

$$\frac{\partial^3\chi(\mathbf{t})}{\partial t^{\alpha_1,m_1}\partial t^{\alpha_2,m_2}\partial t^{\alpha_+,m_+}} \quad \eta^{(\alpha_+,m_+),(\alpha_-,m_-)}\frac{\partial^3\chi(\mathbf{t})}{\partial t^{\alpha_-,m_-}\partial t^{\alpha_3,m_3}\partial t^{\alpha_4,m_4}} =$$
$$\frac{\partial^3\chi(\mathbf{t})}{\partial t^{\alpha_3,m_3}\partial t^{\alpha_2,m_2}\partial t^{\alpha_+,m_+}} \quad \eta^{(\alpha_+,m_+),(\alpha_-,m_-)}\frac{\partial^3\chi(\mathbf{t})}{\partial t^{\alpha_-,m_-}\partial t^{\alpha_1,m_1}\partial t^{\alpha_4,m_4}}$$

for all $m_i, \alpha_i = 0, 1$ and $i = 1, \ldots, 4$, and where the summation convention has been used.

Setting $(\alpha_1, m_1) = (1, 0)$, $(\alpha_2, m_2) = (0, 1)$, and $(\alpha_3, m_3) = (\alpha_4, m_4) = (1, 1)$ in the WDVV equation and plugging in equation (24), we obtain

$$\begin{aligned}\partial_{1,1}^3\Psi = \quad & - \quad \partial_{0,1}^2\partial_{1,0}\Psi\partial_{1,0}\partial_{1,1}^2\Psi \\ & - \quad \partial_{0,1}\partial_{1,0}^2\Psi\partial_{0,1}\partial_{1,1}^2\Psi \\ & + \quad \frac{1}{3}t^{0,1}\partial_{0,1}^2\partial_{1,1}\Psi \\ & + \quad \partial_{0,1}^2\partial_{1,1}\Psi\partial_{1,0}^2\partial_{1,1}\Psi \\ & + \quad (\partial_{0,1}\partial_{1,0}\partial_{1,1}\Psi)^2,\end{aligned}$$

where we have used the shorthand notation

$$\partial_{\alpha,m}^n = \left(\frac{\partial}{\partial t^{\alpha,m}}\right)^n.$$

Together with the Divisor Axiom in Theorem 4.4.1, we obtain the recursion relations for $\beta = 1$ correlators

$$\langle \tau_{0,1}\tau_{1,1}^3\rangle_1 = \frac{1}{3}\langle \tau_{1,0}^2\tau_{1,1}\rangle_1,$$

and, for all $n_1 \geq 2$,

$$\langle \tau_{0,1}^{n_1}\tau_{1,1}^{2n_1+1}\rangle_1 = \frac{n_1}{3}\langle \tau_{0,1}^{n_1-1}\tau_{1,1}^{2n_1-1}\rangle_1.$$

These collectively imply that for all $n_1 \geq 1$,

$$\langle \tau_{0,1}^{n_1}\tau_{1,1}^{2n_1+1}\rangle_1 = \frac{n_1!}{3^{n_1}}\langle \tau_{1,0}^2\tau_{1,1}\rangle_1.$$

Furthermore, the tensor product property implies that

$$\langle \tau_{1,0}^2\tau_{1,1}\rangle_1 = 1.$$

Together with the Divisor Axiom, this determines all of the $\beta = 1$ correlators.

If $\beta \geq 2$ then we obtain the following recursion relation for all $n_1 \geq 0$:

$$\begin{aligned}
& \langle \tau_{0,1}^{n_1}\tau_{1,1}^{6\beta+2n_1-5}\rangle_\beta = \frac{n_1\beta^2}{3}\langle \tau_{0,1}^{n_1-1}\tau_{1,1}^{6\beta+2n_1-7}\rangle_\beta \\
+ \quad & \sum(-\beta'\beta''\binom{n_1}{n_1'}\binom{6\beta+2n_1-8}{6\beta'+2n_1'-1}\langle \tau_{0,1}^{n_1'+2}\tau_{1,1}^{6\beta'+2n_1'-1}\rangle_{\beta'}\langle \tau_{1,0}^{n_1''}\tau_{1,1}^{6\beta''+2n_1''-5}\rangle_{\beta''} \\
- \quad & (\beta')^2\binom{n_1}{n_1'}\binom{6\beta+2n_1-8}{6\beta'+2n_1'-3}\langle \tau_{0,1}^{n_1'+1}\tau_{1,1}^{6\beta'+2n_1'-3}\rangle_{\beta'}\langle \tau_{1,0}^{n_1''+1}\tau_{1,1}^{6\beta''+2n_1''-3}\rangle_{\beta''} \\
+ \quad & (\beta'')^2\binom{n_1}{n_1'}\binom{6\beta+2n_1-8}{6\beta'+2n_1'-2}\langle \tau_{0,1}^{n_1'+2}\tau_{1,1}^{6\beta'+2n_1'-1}\rangle_{\beta'}\langle \tau_{1,0}^{n_1''}\tau_{1,1}^{6\beta''+2n_1''-5}\rangle_{\beta''} \\
+ \quad & \beta'\beta''\binom{n_1}{n_1'}\binom{6\beta+2n_1-8}{6\beta'+2n_1'-4}\langle \tau_{0,1}^{n_1'+1}\tau_{1,1}^{6\beta'+2n_1'-3}\rangle_{\beta'}\langle \tau_{1,0}^{n_1''+1}\tau_{1,1}^{6\beta''+2n_1''-3}\rangle_{\beta''}),
\end{aligned}$$

where the first summation is over $\beta', \beta'' \geq 1$ such that $\beta = \beta' + \beta''$, and over $n_1', n_1'' \geq 0$ such that $n_1 = n_1' + n_1''$. Furthermore, we have defined

$$\langle \tau_{0,1}^{-1}\tau_{1,1}^{6\beta-7}\rangle_\beta := 0.$$

Together with the Divisor Axiom, these recursion relations completely determine all of the n-point correlators of the theory where $n \geq 3$.

Finally, the 0, 1 and 2 point correlators (those in the unstable range) are determined as a special case of Theorem 4.5.3. The only nonvanishing correlators of these types are

$$\langle \tau_{1,1}\rangle_1 = \langle \tau_{1,0}\tau_{1,1}\rangle_1 = 1.$$

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