Math 124, Practice Exam Solutions for Exam #1, February 23, 2000

1. Calculate the following:

(a) Let $u = \sin x$ then $du = \frac{du}{dx}dx = \cos x dx$ and

$$\int \sin^{100}(x)\cos(x)\,dx = \int u^{100}\,du = \frac{u^{101}}{101} + C = \frac{\sin^{101}(x)}{101} + C$$

(b) Let $u = x^3 - 8x^2 + 5x + 3$ then $du = \frac{du}{dx}dx = (3x^2 - 16x + 5)\,dx$ then

$$\begin{aligned}\int \frac{3x^2 - 16x + 5}{\sqrt{x^3 - 8x^2 + 5x + 3}}\,dx &= \int (3x^2 - 16x + 5)(x^3 - 8x^2 + 5x + 3)^{-\frac{1}{2}}\,dx \\ &= \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} + C = 2(x^3 - 8x^2 + 5x + 3)^{\frac{1}{2}} + C\end{aligned}$$

(c) Let $u = x^2 - 3x$ then $\frac{du}{dx} = (2x - 3)$ or $\frac{1}{2}\frac{du}{dx} = (x - \frac{3}{2})\,dx$ and

$$\int (x - \frac{3}{2})\sin(x^2 - 3x)\,dx = \int \sin(u)\,\frac{1}{2}\,du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2 - 3x) + C$$

(d)

$$\int x\ln x^3\,dx = \int x(3\ln x)\,dx = 3\int x\ln x\,dx = 3\int u\,dv$$

where $u = \ln x$ and $v = \frac{x^2}{2}$. Using integration by parts, we obtain

$$\int x\ln x^3\,dx = 3\left(\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{d}{dx}(\ln x)\,dx\right) = \frac{3x^2}{2}\ln x - \frac{3x^2}{4} + C$$

(e) Choose $u = \tan x$ then $du = \sec^2 x\,dx$. Also, when $x = 0$ then $u = \tan 0 = 0$ and when $x = \frac{\pi}{4}$ then $u = \tan\frac{\pi}{4} = 1$ so

$$\int_0^{\frac{\pi}{4}} \tan^5 x \sec^2 x\,dx = \int_0^1 u^5\,dx = \left.\frac{u^6}{6}\right|_0^1 = \frac{1}{6}$$

(f) Let $u = f(x)$ then $du = \frac{du}{dx}dx = f'(x)\,dx$. Therefore,

$$\int_2^4 f'(x)\sin(f(x))dx = \int_{f(2)}^{f(4)} \sin(u)\,du = \int_1^7 \sin(u)\,du = -\cos(7) + \cos(1).$$

(g) Notice that

$$\int (2x - 8)e^{-x}\,dx = 2\int x\,e^{-x}\,dx - 8\int e^{-x}\,dx = 2\int x\,e^{-x}\,dx + 8e^{-x}$$

but by integration by parts,

$$\int x\,e^{-x}\,dx = -e^{-x} - xe^{-x}C$$

therefore,

$$\int (2x - 8)e^{-x}\,dx = 6e^{-x} - 2xe^{-x} + C$$

and

$$\int_1^3 (2x - 8)e^{-x}\,dx = -\frac{4}{e}$$

(h) The average value by definition is

$$\overline{f} = \frac{1}{8-3}\int_3^8 x^2\,dx = \frac{97}{3}.$$

2. First, note that that two graphs intersect when $x^3 - x^5 = 0$ or, equivalently, when $x^3(1-x^2) = 0$ which occurs when $x = 0, \pm 1$. Therefore, since $x \geq 0$, R lies between $x = 0$ and $x = 1$.

   (a) The area, A, of R is

   $$A = \int_0^1 (x^3 - x^5)dx = \frac{1}{12}.$$

   (b) The centroid $(\overline{x}, \overline{y})$ is defined by

   $$\overline{x} = \frac{1}{A}\int_0^1 x(x^3 - x^5)dx = \frac{24}{35}$$

   and

   $$\overline{y} = \frac{1}{A}\int_0^1 \frac{1}{2}(x^3 - x^5)^2 dx = \frac{16}{231}$$

   where the last integral has been evaluated using the fact that $(x^3 - x^5)^2 = x^6 - 2\,x^8 + x^{10}$.

3. The arc length L is given by

   $$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

   but

   $$\frac{dy}{dx} = 3x^{\frac{1}{2}}.$$

   Plugging this into the arc length formula, we need to calculate the integral

   $$L = \int_1^4 \sqrt{1+9x}\,dx$$

   but by using substitution, we can see that

   $$\int \sqrt{1+9x}dx = \frac{2}{27}(1+9x)^{\frac{3}{2}}$$

   Plugging in, we obtain

   $$L = \frac{-20\sqrt{10}}{27} + \frac{74\sqrt{37}}{27}.$$

4. The graphs intersect when $0 = (x-2)^2 - x = x^2 - 5x + 4 = (x-4)(x-1)$; in otherwords, at $x = 1, 4$.

   The volume V of the region is

   $$V = \int_1^4 (\pi x^2 - \pi((x-2)^2)^2)\,dx$$

   but

   $$\int (\pi x^2 - \pi((x-2)^2)^2)\,dx = -16\,\pi\,x + 16\,\pi\,x^2 - \frac{23\,\pi\,x^3}{3} + 2\,\pi\,x^4 - \frac{\pi\,x^5}{5}.$$

   Plugging in the limits, we obtain

   $$V = \frac{72}{5}\pi.$$