Math 124, Practice Exam Solutions for Exam #1, February 23, 2000

1. Calculate the following:
   (a) Let \( u = \sin x \) then \( du = \frac{du}{dx} dx = \cos x \) \( dx \) and
       \[
       \int \sin^{100}(x) \cos(x) \, dx = \int u^{100} \, du = \frac{u^{101}}{101} + C = \frac{\sin^{101}(x)}{101} + C
       \]
   (b) Let \( u = x^3 - 8x^2 + 5x + 3 \) then \( du = \frac{du}{dx} dx = (3x^2 - 16x + 5) \) \( dx \) then
       \[
       \int \frac{3x^2 - 16x + 5}{\sqrt{x^3 - 8x^2 + 5x + 3}} \, dx = \int (3x^2 - 16x + 5)(x^3 - 8x^2 + 5x + 3)^{-\frac{1}{2}} \, dx
       \]
       \[
       = \int u^{-\frac{1}{2}} \, du = 2u^{\frac{1}{2}} + C = 2(x^3 - 8x^2 + 5x + 3)^{\frac{1}{2}} + C
       \]
   (c) Let \( u = x^2 - 3x \) then \( \frac{du}{dx} = (2x - 3) \) or \( \frac{1}{2} \frac{du}{dx} = (x - \frac{3}{2}) \) \( dx \) and
       \[
       \int (x - \frac{3}{2}) \sin(x^2 - 3x) \, dx = \int \sin(u) \frac{1}{2} \, du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2 - 3x) + C
       \]
   (d)
       \[
       \int x \ln x^3 \, dx = \int x(3\ln x) \, dx = 3 \int x \ln x \, dx = 3 \int u \, dv
       \]
       where \( u = \ln x \) and \( v = \frac{x^2}{2} \). Using integration by parts, we obtain
       \[
       \int x \ln x^3 \, dx = 3 \left( \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{d}{dx}(\ln x) \, dx \right) = \frac{3x^2}{2} \ln x - \frac{3x^2}{4} + C
       \]
   (e) Choose \( u = \tan x \) then \( du = \sec^2 x \, dx \). Also, when \( x = 0 \) then \( u = \tan 0 = 0 \) and when \( x = \frac{\pi}{4} \) then \( u = \tan \frac{\pi}{4} = 1 \) so
       \[
       \int \tan^5 x \sec^2 x \, dx = \int_0^1 u^5 \, dx = \frac{u^6}{6} \bigg|_0^1 = \frac{1}{6}
       \]
   (f) Let \( u = f(x) \) then \( du = \frac{du}{dx} \, dx = f'(x) \, dx \). Therefore,
       \[
       \int_2^4 f'(x) \sin(f(x)) \, dx = \int_{f(2)}^{f(4)} \sin(u) \, du = \int_1^7 \sin(u) \, du = -\cos(7) + \cos(1).
       \]
   (g) Notice that
       \[
       \int (2x - 8)e^{-x} \, dx = 2 \int x e^{-x} \, dx - 8 \int e^{-x} \, dx = 2 \int x e^{-x} \, dx + 8e^{-x}
       \]
       but by integration by parts,
       \[
       \int x e^{-x} \, dx = -e^{-x} - xe^{-x} + C
       \]
       therefore,
       \[
       \int (2x - 8)e^{-x} \, dx = 6e^{-x} - 2xe^{-x} + C
       \]
       and
       \[
       \int_1^3 (2x - 8)e^{-x} \, dx = -\frac{4}{e}
       \]
(h) The average value by definition is
\[ \overline{y} = \frac{1}{8 - 3} \int_{3}^{8} x^2 \, dx = \frac{97}{3}. \]

2. First, note that two graphs intersect when \( x^3 - x^5 = 0 \) or, equivalently, when \( x^3(1 - x^2) = 0 \) which occurs when \( x = 0, \pm 1 \). Therefore, since \( x \geq 0 \), \( R \) lies between \( x = 0 \) and \( x = 1 \).

   (a) The area, \( A \), of \( R \) is
   \[ A = \int_{0}^{1} (x^3 - x^5) \, dx = \frac{1}{12}. \]

   (b) The centroid \((\overline{x}, \overline{y})\) is defined by
   \[ \overline{x} = \frac{1}{A} \int_{0}^{1} x(x^3 - x^5) \, dx = \frac{24}{35} \]
   and
   \[ \overline{y} = \frac{1}{A} \int_{0}^{1} \frac{1}{2}(x^3 - x^5)^2 \, dx = \frac{16}{231} \]
   where the last integral has been evaluated using the fact that \( (x^3 - x^5)^2 = x^6 - 2x^8 + x^{10} \).

3. The arc length \( L \) is given by
   \[ L = \int_{1}^{4} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]
   but
   \[ \frac{dy}{dx} = 3x^\frac{5}{2}. \]
   Plugging this into the arc length formula, we need to calculate the integral
   \[ L = \int_{1}^{4} \sqrt{1 + 9x} \, dx \]
   but by using substitution, we can see that
   \[ \int \sqrt{1 + 9x} \, dx = \frac{2}{27} (1 + 9x)^\frac{3}{2} \]
   Plugging in, we obtain
   \[ L = \frac{-20\sqrt{10}}{27} + \frac{74\sqrt{37}}{27}. \]

4. The graphs intersect when \( 0 = (x - 2)^2 - x = x^2 - 5x + 4 = (x - 4)(x - 1) \); in other words, at \( x = 1, 4 \).
   The volume \( V \) of the region is
   \[ V = \int_{1}^{4} \left[ \pi x^2 - \pi((x - 2)^2)^2 \right] \, dx \]
   but
   \[ \int \left( \pi x^2 - \pi((x - 2)^2)^2 \right) \, dx = -16\pi x + 16\pi x^2 - \frac{23\pi x^3}{3} + 2\pi x^4 - \frac{\pi x^5}{5}. \]
   Plugging in the limits, we obtain
   \[ V = \frac{72}{5}\pi. \]