Math 731A1, Final Exam

Please slide under my office door (MCS 234) by Tues 5/7 at 8am.

(1) (10 points) Pick $a \in \mathbb{C}$. Consider the 3-dimensional Lie algebra over $\mathbb{C}$, $L_{(a)}$, with basis $\{X, Y, Z\}$ satisfying

$$[X, Y] = Y; [X, Z] = aZ; [Y, Z] = 0.$$

(a) Is the Lie algebra $L_{(a)}$ nilpotent for any a?
(b) When is $L_{(a)}$ isomorphic to $L_{(a')}$?

(2) (10 points) Prove that if L is nilpotent, the Killing form of L is identically zero.

(3) (10 points) Prove that a Lie algebra L is solvable if and only if $[L, L]$ lies in the kernel of $K^\flat : L \to L^*$ where $K^\flat(X) := K(X, \cdot)$ for all X in L and K is the Killing form.

(4) (10 points) Let L be a simple Lie algebra. Let β and γ be two symmetric, nondegenerate, invariant, bilinear forms on L. Prove that β and γ are proportional. In particular, if any such form is proportional to the Killing form of L.

(5) (10 points) Let T be the set of diagonal matrices in a classical linear Lie algebra L (of type A_l, B_l, C_l, or D_l) over $\mathbb{C}$.
(a) Prove that $T = N_L(T)$ where $N_L(T) := \{X \in L \,|\, [T, L] \subseteq T\}$ is the normalizer of T in L.
(b) Prove that T is a maximal toral subalgebra of L of dimension l.

(6) (20 points) Let Φ be a root system in E. Consider $\Phi^\vee := \{\alpha^\vee \,|\, \alpha \in \Phi\}$ where $\alpha^\vee := 2\alpha/(\alpha, \alpha)$.
(a) Prove that $\Phi^\vee$ is a root system in E (called the dual root system to Φ) whose Weyl group, $\mathcal{W}^\vee$, is naturally isomorphic to the Weyl group of Φ, $\mathcal{W}$.
(b) Prove that for all α, β in Φ,

$$\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$$

(c) If Δ is a base of Φ, prove that $\Delta^\vee := \{\alpha^\vee \,|\, \alpha \in \Delta\}$ is a base of $\Phi^\vee$.
(d) Draw a picture of $\Phi^\vee$ for the cases A_1, A_2, B_2, and G_2.
(e) Prove that each irreducible root system is isomorphic to its dual, except that B_l and C_l are dual to each other.

(7) (10 points) Prove that the Weyl group of a root system Φ is isomorphic to the direct product of the respective Weyl groups of its irreducible components.