Math 731A1, Homework #1
Introduction to Lie Groups and Lie Algebras

(1) (10 points) Let G be a Lie group. Use the fact that the multiplication map is smooth to prove that the map $\iota : G \to G$ taking $\iota(g) := g^{-1}$ is smooth.
(2) (10 points) Calculate the Lie algebra of $PGL(n, \mathbb{R})$ and $PGL(n, \mathbb{C})$.
(3) (20 points) Prove that $O(2n+1)$ is isomorphic to $SO(2n+1) \times \mathbb{Z}_2$ as Lie groups. Prove that $O(2n)$ is diffeomorphic to $X := SO(2n) \times \mathbb{Z}_2$ but is not a Lie group isomorphism. Describe the multiplication that X would inherit from $O(2n)$ (semi-direct product).
(4) (15 points) Consider the linear map $\phi : \mathrm{so}(3) \to \mathbb{R}^3$ defined by

$$\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \mapsto (x, y, z)$$

for all $x, y, z \in \mathbb{R}$. Prove that ϕ is a linear isomorphism satisfying $\phi([X, Y]) = \phi(X) \times \phi(Y)$ where $\times$ denotes the vector cross product in $\mathbb{R}^3$. Furthermore, show that $\phi(AXA^{-1}) = A\phi(X)$ for $A \in \mathrm{SO}(3)$.
(5) (20 points) Prove that the Lie algebras su(2) and so(3) are isomorphic. Prove that there is a Lie group homomorphism $\phi : SU(2) \to SO(3)$ by considering the adjoint representation(s). Show that ϕ is surjective and has kernel $\mathbb{Z}_2$. Show that $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$.
(6) (10 points) Show that the exponential map of $\mathrm{SL}(2, \mathbb{R})$ is not surjective. What values can the trace $\mathrm{Tr}(\exp(A))$ take for $A \in \mathrm{sl}(2, \mathbb{R})$? Calculate the image of the exponential map.
(7) (15 points) Prove that there is a G-equivariant diffeomorphism $\Phi : G/H \to G/K$ if and only if H and K are conjugate in G.