Math 124, Practice Exam Solutions for Exam #1, February 23, 2004

1. Calculate the following:
   
   (a) Let \( u = \sin x \) then \( du = \frac{du}{dx} dx = \cos x dx \) and
   
   \[
   \int \sin^{100}(x) \cos(x) \, dx = \int u^{100} du = \frac{u^{101}}{101} + C = \frac{\sin^{101}(x)}{101} + C
   \]

   (b) Let \( u = x^3 - 8x^2 + 5x + 3 \) then \( du = \frac{du}{dx} dx = (3x^2 - 16x + 5) \, dx \) then
   
   \[
   \int \frac{3x^2 - 16x + 5}{\sqrt{x^3 - 8x^2 + 5x + 3}} \, dx = \int (3x^2 - 16x + 5)(x^3 - 8x^2 + 5x + 3)^{-\frac{1}{2}} \, dx
   = \int u^{-\frac{1}{2}} du = 2u^{\frac{1}{2}} + C = 2(x^3 - 8x^2 + 5x + 3)^{\frac{1}{2}} + C
   \]

   (c) Let \( u = x^2 - 3x \) then \( du = \frac{du}{dx} dx = \frac{1}{2} \frac{d}{dx} (x^2 - 3x) \, dx \) and
   
   \[
   \int (x^2 - 3x) \sin(x^2 - 3x) \, dx = \int \sin(u) \frac{1}{2} du = \frac{1}{2} (-\cos u) + C = -\frac{1}{2} \cos(x^2 - 3x) + C
   \]

   (d)
   
   \[
   \int x \ln x^3 \, dx = \int x(3 \ln x) \, dx = 3 \int x \ln x \, dx = 3 \int u \, dv
   \]

   where \( u = \ln x \) and \( v = \frac{x^2}{2} \). Using integration by parts, we obtain
   
   \[
   \int x \ln x^3 \, dx = 3 \left( \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{d}{dx} (\ln x) \, dx \right) = \frac{3x^2}{2} \ln x - \frac{3x^2}{4} + C
   \]

   (e) Let’s use the substitution method. Let \( \hat{u} = \sqrt{x} \) then \( du = \frac{1}{2} x^{-\frac{1}{2}} \, dx \) or, in other words,
   
   \( dx = 2x^{\frac{1}{2}} \, \hat{u} \, d\hat{u} \)

   then
   
   \[
   \int e^{\sqrt{x}} \, dx = \int e^{\hat{u}} \, d\hat{u} = 2 \int e^{\hat{u}} \, d\hat{u}.
   \]

   Now we use integration by parts by setting \( u = \hat{u} \) and \( v = e^{\hat{u}} \)

   \[
   2 \int e^{\hat{u}} \, d\hat{u} = 2(\hat{u} e^{\hat{u}} - \int e^{\hat{u}} \, d\hat{u}) = 2(\hat{u} e^{\hat{u}} - e^{\hat{u}}) + C = 2\sqrt{x} e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.
   \]

   (f) Choose \( u = \tan x \) then \( du = \sec^2 x \, dx \). Also, when \( x = 0 \) then \( u = \tan 0 = 0 \) and when \( x = \frac{\pi}{4} \) then \( u = \tan \frac{\pi}{4} = 1 \) so
   
   \[
   \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^5 x \sec^2 x \, dx = \int_{0}^{1} u^5 \, dx = \frac{u^6}{6} \bigg|_{0}^{1} = \frac{1}{6}
   \]

   (g) Let \( u = f(x) \) then \( du = \frac{du}{dx} dx = f'(x) \, dx \). Therefore,
   
   \[
   \int_{2}^{4} f'(x) \sin(f(x)) \, dx = \int_{f(2)}^{f(4)} \sin(u) \, du = \int_{1}^{7} \sin(u) \, du = -\cos(7) + \cos(1).
   \]
(h) Notice that
\[
\int (2x - 8)e^{-x} \, dx = 2 \int xe^{-x} \, dx - 8 \int e^{-x} \, dx = 2 \int xe^{-x} \, dx + 8e^{-x}
\]
but by integration by parts,
\[
\int xe^{-x} \, dx = -e^{-x} - xe^{-x} + C
\]
therefore,
\[
\int (2x - 8)e^{-x} \, dx = 2 \int xe^{-x} \, dx + 8e^{-x}
\]
and
\[
\int_1^3 (2x - 8)e^{-x} \, dx = -\frac{4}{e}
\]

(i) \[\int_{-\infty}^{3} \frac{1}{1 + x^2} \, dx = \lim_{t \to -\infty} \int_{t}^{3} \frac{1}{1 + x^2} \, dx = \lim_{t \to -\infty} (\arctan(3) - \arctan(t)) = \arctan(3) - (-\frac{\pi}{2}) = \arctan(3) + \frac{\pi}{2} \]

(j) This is an improper integral since the integrand is undefined when \(x = \pm 2\) since \(x^2 - 4 = (x - 2)(x + 2)\). Therefore,
\[
\int_1^{5} \frac{x}{x^2 - 4} \, dx = \lim_{t \to 2^{-}} \int_{1}^{t} \frac{x}{x^2 - 4} \, dx + \lim_{t \to 2^{+}} \int_{t}^{5} \frac{x}{x^2 - 4} \, dx
\]
where the original integral converges if and only if both of the terms on the right hand side converge. Now,
\[
\lim_{t \to 2^{-}} \int_{1}^{t} \frac{x}{x^2 - 4} \, dx = \lim_{t \to 2^{-}} \frac{1}{2} \ln |t^2 - 4| - \frac{1}{2} \ln 3
\]
but this diverges. Therefore, the original integral diverges.

(k) The integral \(\int_0^1 x^s \ln x \, dx\) is an improper integral. Thus,
\[
\int_0^1 x^s \ln x \, dx = \lim_{t \to 0^+} \int_{t}^{1} x^s \ln x \, dx.
\]
The indefinite integral is evaluated by integration by parts where \(u = \ln x\) and \(dv = x^s \, dx\), or, \(v = \frac{x^{s+1}}{s+1}\). (Notice we have used \(s \neq -1\) here.) Thus,
\[
\int x^s \ln x \, dx = (\ln x) \left(\frac{x^{s+1}}{s+1}\right) - \int \frac{x^{s+1}}{s+1} \ln x = \frac{x^{s+1}}{s+1} \ln x - \frac{x^{s+1}}{(s+1)^2} + C
\]
Therefore, plugging in the limits of integration, we have
\[
\int_0^1 x^s \ln x \, dx = \lim_{t \to 0^+} \frac{-1 + t^{1+s} - (1 + s) \, t^{1+s} \ln(t)}{(1 + s)^2}.
\]
If \(s < -1\) then the limit fails to exist and the original integral diverges. If \(s > -1\) then the limit converges since the right hand side becomes
\[
-\frac{1}{(s+1)^2} = \frac{1}{s+1} \lim_{t \to 0^+} t^{1+s} \ln t
\]
but
\[
\lim_{t \to 0^+} t^{1+s} \ln t = \lim_{t \to 0^+} \frac{\ln t}{t^{-1-s}} = \lim_{t \to 0^+} \frac{\frac{1}{t}}{(-1-s)t^{2+s}} = \lim_{t \to 0^+} \frac{t^{s+1}}{-1 - s} = 0.
\]
where L’Hopital’s rule has been used in the first equality and \(s > -1\) has been used in the last equality.
(l) The average value by definition is
\[ \bar{f} = \frac{1}{8-3} \int_3^8 x^2 \, dx = \frac{97}{3}. \]

2. First, note that the two graphs intersect when \( x^3 - x^5 = 0 \) or, equivalently, when \( x^3(1 - x^2) = 0 \) which occurs when \( x = 0, \pm 1 \). Therefore, since \( x \geq 0 \), \( R \) lies between \( x = 0 \) and \( x = 1 \).

(a) The area, \( A \), of \( R \) is
\[ A = \int_0^1 (x^3 - x^5) \, dx = \frac{1}{12}. \]

(b) The area of an equilateral triangle with a side of length \( s \) is
\[ \frac{1}{2} \left( \frac{\sqrt{3}}{2} s \right)^2 = \frac{s^2 \sqrt{3}}{4}. \]

The length of an edge of the triangle located at the position \( x \) (where \( 0 \leq x \leq 1 \)) is \( x^3 - x^5 \).

Therefore, the area of such a triangle is thus
\[ A(x) = \frac{\sqrt{3}}{4} (x^3 - x^5)^2. \]

The volume of \( S \) is
\[ V = \int_0^1 A(x) \, dx = \int_0^1 \frac{\sqrt{3}}{4} (x^3 - x^5)^2 \, dx = \frac{\sqrt{3}}{4} \int_0^1 (x^6 - 2x^8 + x^{10}) \, dx. \]

Performing the latter integral, one obtains
\[ V = \frac{\sqrt{3}}{4} \left( \frac{1}{7} - 2 \left( \frac{1}{9} \right) + \frac{1}{11} \right) = \frac{2\sqrt{3}}{693}. \]

3. Since \( y = 9 - x^2 = (3 - x)(3 + x) \), \( y = 0 \) when \( x = \pm 3 \). The area of \( R \) is thus
\[ A = \int_{-3}^{3} (9 - x^2) \, dx = 36. \]

Thus
\[ \bar{x} = \frac{1}{A} \int_{-3}^{3} x(9 - x^2) \, dx = 0 \]
where the last equality is because \( x(9 - x^2) \) is an odd function. However,
\[ \overline{y} = \frac{1}{2A} \int_{-3}^{3} (9 - x^2)^2 \, dx = \frac{1}{72} \int_{-3}^{3} (81 - 18x^2 + x^4) \, dx = \frac{18}{5}. \]

Therefore, the centroid of \( R \) is \((\bar{x}, \overline{y}) = (0, \frac{18}{5})\).

4. The arc length \( L \) is given by
\[ L = \int_1^4 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]
but
\[ \frac{dy}{dx} = 3x^2. \]

Plugging this into the arc length formula, we need to calculate the integral
\[ L = \int_1^4 \sqrt{1 + 9x^2} \, dx \]
but by using substitution, we can see that

$$\int \sqrt{1+9x} \, dx = \frac{2}{27}(1+9x)^{3/2}$$

Plugging in, we obtain

$$L = \frac{-20\sqrt{10}}{27} + \frac{74\sqrt{37}}{27}.$$  

5. Notice that \((x, y) = (1, 0)\) when \(t = 0\) and \((x, y) = (0, -e^{\pi})\) when \(t = \pi\) so we are interested in the interval \(0 \leq t \leq \pi\). The arclength \(L\) is

$$L = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

but the product rule yields

$$\frac{dx}{dt} = e^t(\cos t - \sin t)$$

and

$$\frac{dy}{dt} = e^t(\cos t + \sin t).$$

Therefore,

$$\sqrt{(e^t(\cos t - \sin t))^2 + (e^t(\cos t + \sin t))^2} = \sqrt{2e^{2t}(\cos^2 t + \sin^2 t)} = \sqrt{2}e^t$$

and plugging in we get

$$L = \int_0^\pi \sqrt{2}e^t \, dt = \sqrt{2}(e^\pi - 1).$$

6. The graphs intersect when \(0 = (x - 2)^2 - x = x^2 - 5x + 4 = (x - 4)(x - 1)\); in otherwords, at \(x = 1, 4\).

The volume \(V\) of the region is

$$V = \int_1^4 (\pi x^2 - \pi((x - 2)^2)^2) \, dx$$

but

$$\int (\pi x^2 - \pi((x - 2)^2)^2) \, dx = -16\pi x + 16\pi x^2 - \frac{23}{3}x^3 + 2\pi x^4 - \frac{\pi x^5}{5}.$$

Plugging in the limits, we obtain

$$V = \frac{72}{5}\pi.$$ 

7. (a) Just calculate the following:

$$\frac{d}{dx} \left( Ce^{-2x} + \frac{1}{3}e^x \right) + 2 \left( Ce^{-2x} + \frac{1}{3}e^x \right) = -2Ce^{-2x} + \frac{1}{3}e^x + 2Ce^{-2x} + \frac{2}{3}e^x = e^x$$

(b) Just plug into the general solution

$$8 = y(0) = C + \frac{1}{3}$$

then

$$C = 8 - \frac{1}{3} = \frac{23}{3}.$$
8. (a) The equation can be rewritten as

\[ \frac{dy}{dx} = -3 \frac{\cos x}{y^2} \]

which is separable. Therefore, one obtains

\[ \int y^2 \, dy = \int -3 \cos x \, dx \]

or

\[ \frac{y^3}{3} = -3 \sin x + K \]

where \( K \) is a constant. Solving for \( y \), one obtains

\[ y = (C - 9 \sin x)^{\frac{1}{3}}. \]

where \( C \) is a constant.

(b)

\[ 2 = y(\pi) = C^{\frac{1}{3}} \]

Hence, \( C = 8 \). The particular solution is

\[ y = (8 - 9 \sin x)^{\frac{1}{3}}. \]