Math 564, Final Exam, April 28, 2004
Due at my office on May 5, 2004 by 11am
(Please slide under my door if I'm not there)
Prof. Takashi Kimura

Throughout, $\mathbf{R}$ is the set of real numbers, $\mathbf{Q}$ is the set of rational numbers and $\mathbf{Z}$ is the set of integers.

1. (10 points) Chapter 2: Problem 4.
2. (10 points) Let X be a topological space. Prove that any finite union of compact subspaces of X is compact.
3. (10 points) Pick $p \geq 1$ and consider on $\mathbf{R}^2$ the metric

$$d'(x, y) := (|x_1 - y_1|^p + |x_2 - y_2|^p)^{\frac{1}{p}}$$

   for all x, y in $\mathbf{R}^2$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Prove that for all $p \geq 1$, the topology on $\mathbf{R}^2$ arising from d' is nothing more than the usual topology on $\mathbf{R}^2$.
4. (10 points) Define an equivalence relation on the plane $X := \mathbf{R}^2$ as follows: $(x_0, y_0) \sim (x_1, y_1)$ if and only if

$$|x_0| + |y_0| = |x_1| + |y_1|.$$

   Let X^* denote the collection of equivalence classes, in the quotient topology. X^* is homeomorphic to a familiar space. What is it?
5. (10 points) Chapter 3: Problem 31.
6. (10 points) Chapter 4: Problem 7.
7. (10 points) Chapter 5: Problem 11.
8. (10 points) Chapter 5: Problem 24.
9. (10 points) Let K be the 1-dimensional simplicial complex consisting of the edges and vertices of a tetrahedron. Let v be a vertex of K. Let L be the subcomplex of K containing the vertex v consisting of all the edges attached to v. Calculate $\pi_1(|K|, v)$ in terms of generators and relations as we did for the Klein bottle on page $136 - 137$ of the text.
10. (10 points) Let $SU(2)$ denote the group of 2×2 complex matrices A which have unit determinant and which satisfy the equation $A^\dagger = A^{-1}$ where $A^\dagger := \overline{A^T}$, A^T denotes the transpose of A, and $\overline{B}$ is the matrix B but where each entry has been replaced by its complex conjugate.
    (a) Prove that $SU(2)$ is a topological group where the group multiplication is matrix multiplication.
    (b) Prove that $SU(2)$ is homeomorphic to S^3.
    (c) Consider the subgroup $G := \{ \pm\mathbf{1} \}$ of $SU(2)$ where $\mathbf{1}$ denotes the identity matrix then we have the action of the group G on $SU(2)$ taking

$$G \times SU(2) \to SU(2)$$

    defined by $(\pm\mathbf{1}, A) \mapsto \pm A$. What is the orbit space $SU(2)/G$ homeomorphic to?
    (d) Find the fundamental group of $SU(2)/G$.