Math 722A1, Final Exam
Differential Topology II

Due Monday May 9, 2005 at noon.

Please slide exams under my office door at MCS 234.

(1) Spivak Vol 1, Chapter 10, Problem 12
(2) Spivak Vol 1, Chapter 10, Problem 26
(3) Let M be a smooth manifold with a Koszul connection ∇.
  (a) For any α in $\Omega^1(M)$, define ∇' via
$$\nabla'_X Y := \nabla_X Y + \alpha(X) Y$$
  for all vector fields X, Y on M. Prove that ∇' is also a Koszul connection.
  (b) How are the curvatures and torsions of both ∇ and ∇' related?
(4) Let $\mathbf{g}$ be a Lie algebra. Let $\mathrm{ad} : \mathbf{g} \to \mathrm{End}(\mathbf{g})$ be defined by $\mathrm{ad}(X)Y := [X, Y]$ for all X, Y in $\mathbf{g}$. Let K be the bilinear form on $\mathbf{g}$ defined by
$$K(X, Y) := \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$$
where tr denotes the trace over $\mathbf{g}$.
  (a) Prove that for all X and Y in $\mathbf{g}$,
$$\mathrm{ad}([X, Y]) = [\mathrm{ad}(X), \mathrm{ad}(Y)]$$
  where the right hand side is the commutator.
  (b) Prove that K is a symmetric, bilinear form satisfying
$$K(X, [Y, Z]) = K([X, Y], Z)$$
  for all X, Y, Z in $\mathbf{g}$.
  (c) Prove that if K is nondegenerate then $\mathbf{g} = [\mathbf{g}, \mathbf{g}]$ where $[\mathbf{g}, \mathbf{g}]$ is the subspace of $\mathbf{g}$ generated by the vectors of the form $[A, B]$ for all A, B in $\mathbf{g}$.
  (d) Let (M, ω) be a symplectic manifold with the action of a Lie group G which preserves the symplectic structure. Prove that if K on the associated Lie algebra $\mathbf{g}$ is nondegenerate then the symplectic vector field $\widetilde{Y}$ on M associated to any element Y in $\mathbf{g}$ is a Hamiltonian vector field.
(5) Let $M := \mathbb{R}^2/\mathbb{Z}^2$. Let $dx \wedge dy$ denote the 2-form on $\mathbb{R}^2$ in cartesian standard coordinates and let ω denote the symplectic form on M induced from $dx \wedge dy$.
  (a) Prove that (M, ω) is a symplectic manifold.
  (b) Consider the Lie group $G := \mathbb{R}^2$. Consider the group action $G \times M \to M$ defined by
$$((a, b), [x, y]) \mapsto [x + a, y + b]$$
  for all (a, b) in G and $[x, y]$ in M. Prove that this group action preserves the symplectic structure but the symplectic vector field $\widetilde{Y}$ on M associated to any element Y in the Lie algebra $\mathbf{g}$ associated to G need not be a Hamiltonian vector field.
(6) Let (M, ω) be a compact, connected, symplectic manifold with the action of a Lie group G which preserves the symplectic structure. Furthermore, suppose that $\omega = d\theta$ where θ is a G-invariant 1-form on M. Prove that the action of G on (M, ω) is Hamiltonian.