Math 731A1, Final Exam
Lie Groups, Lie Algebras, and Representations
Due Wednesday May 11, 2005 at noon.
Please slide exams under my office door at MCS 234.

(1) Let G be a group which is also a manifold with a smooth multiplication $\mu : G \times G \to G$ Prove that the map $\iota : G \to G$ taking $\iota(g) := g^{-1}$ is smooth.
(2) Let G be a Lie group and G_0 be the connected components containing the identity. Prove that G_0 is a normal subgroup of G.
(3) Calculate the Lie algebra of $PGL(n, \mathbb{R})$.
(4) Describe a method using characters to compute the multiplicity of V_i in $\Lambda^k V_n$ where V_l denotes the irreducible representation of $SU(2)$ of dimension $(l+1)$.
(5) Let G be a compact, connected Lie group and let R be its set of real roots. For all α in R, there are elements X_α in L_α, $X_{-\alpha}$ in $L_{-\alpha}$, and H_α in $LT_{\mathbb{C}}$ such that
$$[H_\alpha, X_{\pm\alpha}] = \pm 2 X_{\pm\alpha}$$
and
$$[X_\alpha, X_{-\alpha}] = -H_\alpha.$$
Hence, $\{ X_{\pm\alpha}, H_\alpha \}$ generate a Lie subalgebra of $LG_{\mathbb{C}}$ isomorphic to $sl(2, \mathbb{C})$ for all α in R.
(6) Verify that the only root systems of rank 2 are those given on page 200 of the xeroxed text.
(7) Let α and β be roots in a basis and let s_α and s_β be the associated reflections. Show that $s_\alpha s_\beta$ has order $n = 2, 3, 4$ or 6 according to whether the angle between α and β is $\pi/2$, $2\pi/3$, $3\pi/4$ or $5\pi/6$.
(8) Prove that a root system splits into a sum of irreducible root systems. The subspaces $V_\nu \subset V$ and the subsets $R_\nu \subset R$ of such a splitting are uniquely determined by $R \subset V$. Show that the Weyl group of R is the direct product of the Weyl groups of the irreducible components of R.
(9) Let R be an irreducible root system in V. Show that up to a constant factor, there is exactly one W-invariant Euclidean metric on V.
(10) Construct the root system of G_2 by starting only with the Dynkin diagram of G_2 (see page 212 of the xeroxed text).
(11) Let $\mathbf{g}$ be a Lie algebra of dimension n. Consider $\Lambda^\bullet \mathbf{g}^*$ together with the map $d : \Lambda^k \mathbf{g}^* \to \Lambda^{k+1} \mathbf{g}^*$ for all k given by the formula
$$(d\omega)(X_1, \ldots, X_{k+1}) := -\sum_{1 \le i < j \le n} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1})$$
where $\widehat{X}_i$ and $\widehat{X}_j$ means remove those vectors from the arguments.
   (a) Prove that $d^2 = 0$.
   (b) For all $k \ge 0$, the *k-th Lie algebra cohomology* is
   $$H^k_{\mathrm{Lie}}(\mathbf{g}) := \{ d\omega = 0 \,|\, \omega \in \Lambda^k \mathbf{g}^* \} / d\Lambda^{k-1} \mathbf{g}^*$$
   where $\Lambda^{-1}\mathbf{g}^* := 0$. Prove that the wedge product on $\Lambda^\bullet \mathbf{g}^*$ induces a multiplication on $H^\bullet_{\mathrm{Lie}}(\mathbf{g})$ which is unital, commutative (up to sign) and associative.
   (c) Prove that if G is a Lie group then there is a ring homomorphism $H^\bullet_{\mathrm{Lie}}(\mathbf{g}) \to H^\bullet(G)$ where $H^\bullet(G)$ is the de Rham cohomology of G.