MATHEMATICS 731 A1
An Introduction to Lie Groups and Lie Algebras
Spring Semester 2005

Instructor: Takashi Kimura
e-mail: `kimura@math.bu.edu`
Phone: (617)353-1486 **Office:** MCS 234

Lectures: MWF 1-2
Text: *Lie Groups, Lie Algebras, and Representations, An Elementary Introduction*, by B. Hall, ISBN: 0-387-40122-9, Spring-Verlag.
Supplemental Text:: *Lie Groups, Lie Algebras and Their Representations*, by V. S. Varadarajan, ISBN: 0387909699, Springer Verlag
Supplemental Text:: *Introduction to Lie Algebras and Representation Theory*, by J. E. Humphreys, ISBN: 72-85951, Springer Verlag
My Office Hours: MW 11-12, F 12-1
Class Web Page: `http://math.bu.edu/people/kimura/Spring05/731/`
Content: Lie groups and Lie algebras form an elegant subject which lies at the interface of geometry and algebra. It has many important applications to algebra, geometry, partial differential equations, and quantum field theory – indeed, to any system possessing smooth symmetries. This course will study such symmetries and particularly their actions of vector spaces.

A group G is a set together with an associative multiplication, $m : G \times G \to G$, such that G has an identity element, $\mathbf{1}$, and each element in G has an inverse. A smooth n-dimensional manifold is a space which is obtained by smoothly gluing together open subsets of $\mathbf{R}^n$. A Lie group is a group G which is also a smooth manifold that has the property that the multiplication m is a smooth map. The primary source of examples of Lie groups are groups of matrices. The set of all $n \times n$ invertible matrices with $\mathbf{R}$ entries, $GL(n, \mathbf{R})$, is a Lie group since $GL(n, \mathbf{R})$ is an n^2 dimensional manifold (in fact, it's an open submanifold of $\mathbf{R}^{n^2}$) and matrix multiplication is a smooth map.

Lie groups typically arise as symmetries of spaces or of a system of equations. Consider $SO(n)$, the Lie group consisting of the set of all $n \times n$ matrices whose transpose is equal to its inverse. $SO(n)$ is called the special orthogonal group. This group acts linearly upon elements in $\mathbf{R}^n$ (regarded as column vectors) by matrix multiplication so we obtain a (smooth) map $SO(n) \to \mathrm{End}(\mathbf{R}^n)$. Indeed, $SO(n)$ is nothing more than the group of rotations about the origin in $\mathbf{R}^n$. When a group, *e.g.* $SO(n)$, acts linearly upon a vector space, *e.g.* $\mathbf{R}^n$, this structure is called a representation of the group G.

A Lie algebra is a vector space $\mathbf{g}$ together with a bilinear map $\mathbf{g} \times \mathbf{g} \to \mathbf{g}$ taking $(X, Y) \mapsto [X, Y]$ such that $[X, Y] = -[Y, X]$ and $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ for all X, Y, Z in $\mathbf{g}$. The tangent space at $\mathbf{1}$ to a Lie group G is a Lie algebra. In the example of the rotation group $SO(n)$, one can regard its associated Lie algebra $so(n)$ as infinitesimal rotations about the origin in $\mathbf{R}^n$.

In this course, we will study the relationship between Lie groups and Lie algebras and study their representations. We will cover Lie groups of matrices and their associated Lie algebras, simple Lie algebras, universal enveloping algebras, root systems, Dynkin diagrams, and the classification of complex simple Lie algebras. We will also discuss highest weight representations, weight systems and characters, and the representation ring.

Prerequisites: You should be comfortable with the definition of a group and it will be helpful (although not necessary) to know what a smooth manifold is. All other material will be introduced as necessary.

Homework: Homework will be assigned periodically. Late homework will not be accepted. Students may discuss homework with each other (and are encouraged to do so) but all written work must be prepared independently.

Exams: There will be a final exam.

Grades: Your final grade is determined by – the homeworks, and the final. Grades are based upon the formula:

$$\text{Final Grade} = \frac{1}{2}(\text{ Homework Average}) + \frac{1}{2}(\text{Final Exam})$$