Math 563A1, Final Exam, December 6, 2006
Prof. Takashi Kimura

This exam is due on Wednesday Dec 13 at 5pm in my office MCS 234. Please slide it under my door if I'm not there.

The exam is open book and you may work with others although you must write up the solutions yourself. Good luck!

1. (10 points) Calculate the first fundamental form of a plane

   (a) in Cartesian coordinates, and

   (b) in polar coordinates.

   Verify by direct calculation that the Gaussian curvature (using the expression in Corollary 10.2) is the same whether calculated in one coordinate system or the other.

2. (20 points) Let $f(z)$ and $g(z)$ be the functions in Weierstrass' representation of a minimal surface.

   (a) Prove that for all real numbers t, the family of minimal surfaces associated to functions $e^{it}f(z)$ and $g(z)$ are (locally) isometric, *i.e.* they have the same first fundamental forms. Notice that as t varies from 0 to $\pi/2$, the family goes from a minimal surface to its conjugate.

   (b) Let $f(z) = 1$ and $g(z) = z$. Identify this minimal surface. What happens to this family of minimal surfaces $f(z) = e^{it}$ and $g(z) = z$ as t goes from 0 to $\pi/2$?

3. (10 points) Let S be a compact, orientable, regular surface in $\mathbf{R}^3$ not homeomorphic to the sphere. Prove that there exists points on S where the Gaussian curvature is positive, is zero, and is negative.

4. (20 points) Consider the torus of revolution generated by rotating the circle

$$(x-a)^2 + z^2 = r^2$$

   and

$$y = 0$$

   around the z axis ($a > r > 0$). The parallels generated by the points $(a+r, 0)$ $(a-r, 0)$, and (a, r) are called the maximum parallel, minimum parallel, and the upper parallel, respectively. Check which of these parallels is

   (a) A geodesic.

   (b) An asymptotic curve.

   (c) A line of curvature.

Furthermore, find the geodesic curvature of the upper parallel. (See Exercise 6.18 for the definition of lines of curvature and Exercise 6.12 for the definition of asymptotic curves.)

5. (10 points) Let Σ be a surface without umbilic points and whose Weingarten matrix never vanishes. Let $x(u,v)$ be a surface chart of Σ such that its coordinate curves are lines of curvature. Let $U(u,v)$ be its associated unit normal. Consider

$$y(u,v) = x(u,v) + \rho_1 U(u,v)$$

and

$$z(u,v) = x(u,v) + \rho_2 U(u,v)$$

where $\rho_1 := 1/k_1$ and $\rho_2 := 1/k_2$ where k_1 and k_2 are the principal curvatures of Σ. The surfaces parametrized by $y(u,v)$ and $z(u,v)$ (call them Σ' and Σ'' resp.) are called the *focal surfaces* of Σ. Prove that

If $\frac{\partial k_1}{\partial u}$ and $\frac{\partial k_2}{\partial u}$ are nowhere zero then y and z are surface charts of (some open domain of) Σ' and Σ'' resp.