**Math 563A1, Take Home Midterm Exam, October 20, 2006**
**Prof. Takashi Kimura**

This exam is due at 5pm at my office MCS 234 on Thursday October 26. (Please slide it under my door if I'm not around.) The exam is open book and you may work with others although you must write up the solutions yourself. Good luck!

1. (10 points) Let $\alpha : I \to \mathbf{R}$ be a regular curve parametrized by arc length whose curvature is nowhere vanishing. Let

$$x(s,v) = \alpha(s) + r\,(n(s)\cos(v) + b(s)\sin(v))$$

(where $r$ is a nonzero constant and $s$ belongs to $I$) be a parametrized surface (the *tube* of radius $r$ around $\alpha$), where $n$ is the normal vector and $b$ is the binormal vector of $\alpha$. Show that if $x$ is regular then its unit normal vector is

$$N(s,v) = -(n(s)\cos(v) + b(s)\sin(v)).$$

2. (25 points) Define a surface $S$ as the image of the map

$$\sigma(u,v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$$

for all $(u,v)$ such that $u^2 + v^2 < 3$.

   (a) Prove that $\sigma$ is a regular, single surface patch.
   Hint: Use polar coordinates $u = r\cos\theta$ and $v = r\sin\theta$ and show that

$$x^2 + y^2 + \frac{4}{3}z^2 = \frac{1}{9}r^2(3 + r^2)^2$$

   holds where $(x,y,z)$ belongs to $S$. Then show that the equality implies that points in the $(u,v)$-plane on different circles about $(0,0)$ cannot be mapped to the same point.

   (b) Calculate the first fundamental form of $\sigma$.

   (c) Calculate the second fundamental form of $\sigma$.

   (d) Find the principal curvatures and principal vectors of $S$.

   (e) Let $D$ denote subset of $S$ consisting of those points $\sigma(u,v)$ such that $(u,v)$ belongs to the set $[-1,1] \times [0,1]$. Find the area of $D$.

3. (10 points) Let $\sigma(u,v) := (R\cos u\cos v, R\sin u\cos v, R\sin v)$, where $0 < u < 2\pi$ and $-\frac{\pi}{2} < v < \frac{\pi}{2}$, denote a surface patch for the sphere of radius $R$. Denote the surface that is the image of $\sigma$ by $S$. Let $\tau(r,s) := (r,s,0)$ be the surface patch for a surface $T := \{(r,s,0) \,|\, 0 < r < 2\pi, s \in \mathbf{R}\}$. Consider the map $I : S \to T$ taking

$$I(\sigma(u,v)) := (u, \ln\tan(\frac{v}{2} + \frac{\pi}{4}), 0).$$

Show that $I$ is a conformal map.

4. (15 points) Prove the following statements.

   (a) If a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

   (b) The sum of the normal curvatures for any pair of orthogonal directions at a point $p$ in $S$ is constant.