Math 563A1, Final Exam, December 10, 2007
Prof. Takashi Kimura

This exam is due on Tuesday Dec 18 at noon in my office MCS 234. Please slide it under my door if I'm not there.

The exam is open book and you may work with others although you must write up the solutions yourself. Good luck!

1. (10 points) Prove the following statements.
   (a) Let S be a surface with everywhere non-positive Gaussian curvature then no closed geodesic bounds a simply connected region.
   (b) Let S be a compact, orientable, regular surface in $\mathbf{R}^3$ not homeomorphic to the sphere then there exists points on S where the Gaussian curvature is positive, is zero, and is negative.

2. (10 points) Consider the torus of revolution generated by rotating the circle
$$(x-a)^2 + z^2 = r^2$$
in the xz-plane around the z axis ($a > r > 0$). The parallels generated by the points $(a+r, 0)$ $(a-r, 0)$, and (a, r) are called the maximum parallel, minimum parallel, and the upper parallel, respectively. Check which of these parallels is
   (a) A geodesic.
   (b) An asymptotic curve.
   (c) A line of curvature.

   Furthermore, find the geodesic curvature of the upper parallel. (See Exercise 6.18 for the definition of lines of curvature and Exercise 6.12 for the definition of asymptotic curves.)

3. (10 points) Let $\alpha : I \to S$ be a curve parametrized by its arc length s which has everywhere nonzero curvature. Consider the parametrized surface
$$\sigma(s, v) := \alpha(s) + vb(s)$$
where (s, v) belongs to $I \times (-\epsilon, \epsilon)$ for $\epsilon > 0$, I an open interval, and b is the binormal vector of α. Prove that if ϵ is small then image of σ, S, is a regular surface and that α is a geodesic of S.

4. (10 points) Prove that there are no compact minimal surfaces.

5. (10 points) Prove the following statements.
   (a) If a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.
   (b) The sum of the normal curvatures for any pair of orthogonal directions at a point p in S is constant.