Math 563A1, Homework 1
Due September 24, 2007
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1. (10 points) Let a, b, c belong to $\mathbf{R}^3$. Prove the following identities:

   (a)
$$a \cdot (b \times c) = -b \cdot (a \times c)$$

   (b)
$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

2. (10 points) Consider the parametrized plane curve $\gamma : \mathbf{R} \to \mathbf{R}^2$ given by $\gamma(t) := (t, \cosh t)$. This curve is called a catenary.

   (a) Find the velocity (i.e. the tangent vector) and speed of γ.

   (b) Find the curvature κ of γ.

   (c) Find the signed curvature[1] κ_s of γ.

3. (10 points) Let γ be a regular curve in $\mathbf{R}^n$ such that all of its tangent lines pass through a fixed point. Show that γ is (a part of) a straight line.

4. (10 points) Consider the curve
$$\beta(s) = (\frac{1}{3}(1+s)^{\frac{3}{2}}, \frac{1}{3}(1-s)^{\frac{3}{2}}, \frac{s}{\sqrt{2}})$$
   where $-1 < s < 1$.

   (a) Show that β has unit speed.

   (b) Show that
$$\kappa = \frac{1}{\sqrt{8(1-s^2)}}.$$

   (c) Show that
$$N = (\sqrt{\frac{1-s}{2}}, \sqrt{\frac{1+s}{2}}, 0)$$
   and
$$B = T \times N = (-\frac{1}{2}\sqrt{1+s}, \frac{1}{2}\sqrt{1-s}, \frac{1}{\sqrt{2}}).$$

   (d) Show that $\tau = \kappa$.

[1]The signed curvature of a nonunit speed curve is, by definition, the signed curvature of a unit speed reparametrization of the curve.

5. (10 points) If a rigid body moves along a (unit speed) curve $\alpha(s)$ then the motion of the body consists of a translation along α and rotation about α. The rotation is determined by an angular velocity vector ω, called the Darboux vector, which satisfies the equations $T' = \omega \times T$, $N' = \omega \times N$, and $B' = \omega \times B$.

   (a) Show that ω, in terms of T, N, and B is given by $\omega = \tau T + \kappa B$. Hint: Write $\omega = aT + bN + cB$ and take cross products with T, N, and B to determined a, b, c.

   (b) Show that $T' \times T'' = \kappa^2 \omega$.