Math 563A1, Take Home Midterm Exam, October 19, 2007
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This exam is due in class on Friday October 26. The exam is open book and you may work with others although you must write up the solutions yourself. Good luck!

1. (10 points) Show that for all a, b, u, v in $\mathbf{R}^3$, the following identity holds:

$$(v \times w) \cdot (a \times b) = (v \cdot a)(w \cdot b) - (v \cdot b)(w \cdot a).$$

2. (10 points) Let α be a unit-speed curve which lies on a sphere of radius R with center located at p in $\mathbf{R}^3$. Show that if $\tau \neq 0$ then

$$\alpha(s) - p = -\frac{1}{\kappa}N - \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}B$$

and

$$R^2 = \left(\frac{1}{\kappa}\right)^2 + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)^2.$$

On the other hand, show that if $\left(\frac{1}{\kappa}\right)' \neq 0$ and $\left(\frac{1}{\kappa}\right)^2 + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)^2$ is constant then a (unit speed) curve α lies on a sphere.

3. (20 points) Define a surface S as the image of the map

$$\sigma(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$$

for all (u, v) such that $u^2 + v^2 < 3$.

(a) Prove that σ is a regular, single surface patch for S.
Hint: Use polar coordinates $u = r\cos\theta$ and $v = r\sin\theta$ and show that

$$x^2 + y^2 + \frac{4}{3}z^2 = \frac{1}{9}r^2(3 + r^2)^2$$

holds where (x, y, z) belongs to S. Then show that the equality implies that points in the (u, v)-plane on different circles about $(0, 0)$ cannot be mapped to the same point.

(b) Calculate the first fundamental form of σ.

(c) Calculate the second fundamental form of σ.

(d) Consider the path $\gamma(t) = \sigma(r\cos t, r\sin t)$ for any constant $0 < r < \sqrt{3}$.

i. Find r so that γ is unit speed.

ii. For that value of r, calculate the normal curvature κ_n and geodesic curvature κ_g of γ.

(e) Let D denote subset of S consisting of those points $\sigma(u, v)$ such that (u, v) belongs to the set $[-1, 1] \times [0, 1]$. Find the area of D.

4. (10 points) Let $\sigma(u,v) := (R\cos u\cos v, R\sin u\cos v, R\sin v)$, where $0 < u < 2\pi$ and $-\frac{\pi}{2} < v < \frac{\pi}{2}$, denote a surface patch for the sphere of radius R. Denote the surface that is the image of σ by S. Let $\tau(r,s) := (r,s,0)$ be the surface patch for a surface $T := \{(r,s,0) \,|\, 0 < r < 2\pi, s \in \mathbf{R}\}$. Consider the map $I : S \to T$ taking

$$I(\sigma(u,v)) := (u, \ln\tan(\frac{v}{2} + \frac{\pi}{4}), 0).$$

Show that $I \circ \sigma$ is a conformal map.