Math 123, Practice Exam Solutions for Exam #1, October 13, 2008

1. Calculate the following:

(a)

$$\lim_{x\to -1} \frac{x^2-4x-5}{x+1} = \lim_{x\to -1} \frac{(-5+x)(1+x)}{x+1} = \lim_{x\to -1}(-5+x) = -6$$

(b)

$$\lim_{x\to 3} \frac{5x^2}{2x-1} = \frac{5(3)^2}{2(3)-1} = 9$$

(c)

$$\lim_{x\to 2^+} \frac{4-x^2}{|2-x|} = \lim_{x\to 2^+} \frac{4-x^2}{-(2-x)} = \lim_{x\to 2^+} \frac{(2-x)(x+2)}{-(2-x)} = \lim_{x\to 2^+} -(x+2) = -4$$

(d)

$$\lim_{x\to 1} \sqrt{\frac{2x^3-3x+5}{2-x}} = \sqrt{\frac{2\cdot 1^3 - 3\cdot 1 + 5}{2-1}} = \sqrt{4} = 2$$

(e)

$$\lim_{x\to 3} f(x) \quad \text{where} \quad f(x) = \begin{cases} x^2 & \text{if } x > 3 \\ 8 & \text{if } x = 3 \\ 12 - x & \text{if } x < 3 \end{cases}$$

Notice that $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(12-x) = 9$ and $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^2 = 9$. Since both one-sided limits agree with the value 9, $\lim_{x\to 3} f(x) = 9$.

(f)

$$\lim_{h\to 0} \frac{\sqrt[3]{8+h}-2}{h} = \lim_{h\to 0} \frac{\sqrt[3]{8+h}-\sqrt[3]{8}}{h} = f'(8)$$

where $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ by the definition of derivative. Hence $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ and $f'(8) = \frac{1}{3}(8^{-\frac{2}{3}}) = 1/12$.

(g)

$$\lim_{x\to 3^-} \frac{x+3}{x^2-9} = \lim_{x\to 3^-} \frac{x+3}{(x-3)(x+3)} = \lim_{x\to 3^-} \frac{1}{x-3} = -\infty$$

(h) The horizontal and vertical asymptotes of

$$y = \frac{4-3x}{\sqrt{16x^2+1}}$$

Since $\lim_{x\to\infty} \frac{4-3x}{\sqrt{16x^2+1}} = -\frac{3}{4}$ and $\lim_{x\to -\infty} \frac{4-3x}{\sqrt{16x^2+1}} = \frac{3}{4}$, the horizontal asymptotes are $y = \frac{3}{4}$ and $y = -\frac{3}{4}$. There are no vertical asymptotes since $\frac{4-3x}{\sqrt{16x^2+1}}$ exists for all x values.

(i) $f'(x)$ where

$$f(x) = \sin\left(x^{100}\right)$$

Applying the chain rule, we obtain

$$f'(x) = 100x^{99}\cos\left(x^{100}\right)$$

(j) $f'(x)$ where

$$f(x) = \sqrt{e^{2x} + 7x}$$

Applying the chain rule, we obtain

$$f'(x) = \frac{1}{2}(e^{2x} + 7x)^{-\frac{1}{2}}(2e^{2x} + 7).$$

(k) $f'(x)$ where

$$f(x) = 10^{\cos x}$$

Rewrite $10^{\cos x} = e^{\cos x \ln 10}$ then applying the chain rule, we obtain

$$f'(x) = e^{\cos x \ln 10}(-\sin x \ln 10) = -(\sin x \ln 10)10^{\cos x}.$$

(l) $f'(x)$ where

$$f(x) = \frac{\ln x}{x}$$

Use the quotient rule to obtain

$$f'(x) = \frac{x\frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

(m) $f'(x)$ where

$$f(x) = x^{x^2}$$

First, rewrite as $f(x) = \exp\left(x^2 \ln x\right)$ then

$$f'(x) = \exp\left(x^2 \ln x\right)\left(2x\ln x + x^2\frac{1}{x}\right) = x^{x^2}\left(2x\ln x + x\right)$$

(n) $f'(x)$ where

$$f(x) = \arctan(x^3)$$

$$f'(x) = \frac{1}{1 + (x^3)^2}3x^2 = \frac{3x^2}{1 + x^6}$$

2. Consider the function

$$f(x) = \begin{cases} x - c, & \text{if } x > 2; \\ 3x^2, & \text{if } x \leq 2 \end{cases}$$

where c is a real number.

(a) What value of c makes the function f continuous everywhere? The only place where the function f might not be continuous is at $x = 2$ but we can avoid this if we choose c so that the graphs of $y = x - c$ and $y = 3x^2$ agree at $x = 2$, *i.e.* so there is no jump across $x = 2$. Solving $2 - c = 3 \cdot 2^2$ means that $c = -10$.

(b) If $x > 2$ then $f(x) = x - c \Rightarrow f'(x) = 1 \Rightarrow f'(7) = 1$

(c) If $x < 2$ then $f(x) = 3x^2 \Rightarrow f'(x) = 6x \Rightarrow f'(-1) = -6$

(d) Call $f_1(x) = x - c$ and $f_2(x) = 3x^2$. Notice that $f'(2)$ does not exist (even when $c = -10$) since $f_1'(2) = 1$ which does not agree with $f_2'(2) = 6 \cdot 2 = 12$.

3. Compute the derivative of the following functions:

(a)

$$f(x) = 3x^5 - x^2 + 9 \Rightarrow f'(x) = 15x^4 - 2x$$

(b)

$$f(x) = \frac{2}{x^2} - 3\sqrt{x} = 2x^{-2} - 3x^{\frac{1}{2}} \Rightarrow f'(x) = -4x^{-3} - \frac{3}{2}x^{-\frac{1}{2}}$$

(c)

$$f(x) = \frac{x^2}{2x - 3} \Rightarrow$$

$$f'(x) = \frac{(2x-3)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(2x-3)}{(2x-3)^2} = \frac{(2x-3)(2x) - x^2(2)}{(2x-3)^2} = \frac{2x^2 - 6x}{(2x-3)^2}$$

(d)

$$f(x) = x^2e^x \Rightarrow f'(x) = e^x\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(e^x) = 2xe^x + x^2e^x$$

(e)

$$f(t) = t\sin t \Rightarrow f'(t) = t\frac{d}{dt}(\sin t) + \sin t\frac{d}{dt}(t) = t\cos t + \sin t$$

4. Find the equation for the tangent line to the curve $y = f(x)$ where through the point $(1, -3)$ where

$$f(x) = x^8 - 4x.$$

$f'(x) = 8x^7 - 4$. The slope of the tangent line is $f'(1) = 4$. Therefore the equation for this tangent line is

$$\frac{y - (-3)}{x - 1} = 4$$

or, solving for y, $y = 4x - 7$.

5. Suppose an object is moving along the real line with its position at time t given by the function $s(t) = \frac{1}{3}t^3 - 3t^2 - 7t + 10$.

(a) The object is at rest when the velocity $v(t) = 0$ but $v(t) = s'(t) = t^2 - 6t - 7 = (t-7)(t+1) = 0$. The solution is $t = -1$ or $t = 7$.

(b) The object is moving to the right whenever $(t-7)(t+1) = v(t) > 0$, i.e. it is moving to the right if $t > 7$ or $t < -1$.

(c) The acceleration $a(t) = v'(t) = 2t - 6 > 0$ whenever $t > 3$.

(d) $v(2) = -15$.

6. Find the equation for the tangent line to the curve given by the equation $\cos(xy) - 3y^3 = e^x + 1$ through the point $(0, -1)$.

We need to find $\frac{dy}{dx}$. Use implicit differentiation to obtain

$$\frac{d}{dx}(\cos(xy) - 3y^3) = \frac{d}{dx}(e^x + 1)$$

which yields

$$-\sin(xy)(x\frac{dy}{dx} + y) - 9y^2\frac{dy}{dx} = e^x$$

Now, solve to obtain

$$\frac{dy}{dx} = -\frac{e^x + y\sin(xy)}{9y^2 + x\sin(xy)}$$

At the point $(x, y) = (0, -1)$, $\frac{dy}{dx} = -\frac{1}{9}$ so the equation for the tangent line is

$$\frac{y - (-1)}{x - 0} = -\frac{1}{9}$$

or $y = -1 - \frac{1}{9}x$.

7. Consider the graph of $y = f(x)$ on the next page (figure 1).

(a) Where is f undefined?

$$x = -1, 0, 3$$

(b) Where is f not continuous?

$$x = -5, -1, 0, 3$$

(c) Where is f not differentiable?

$$x = -5, -3, -2, -1, 0, 3, 5$$

(d) On what interval(s) is f' positive? Where does f' vanish? f' is positive on the intervals $(-5, -4)$, $(-2, -1)$, $(2, 3)$, $(3, 5)$. f' vanishes at $x = -4, 2$

(e) On what interval(s) is f concave down? $(-\infty, -5)$,$(-5, -3)$, $(-1, 0)$, $(3, 5)$.

(f) What are

i.

$$f'(6) = \frac{0 - 3}{7.25 - 5} = -\frac{3}{2.25}$$

ii.

$$\lim_{x \to -5^+} f(x) = -2$$

iii.

$$\lim_{x \to -1} f(x) = -1$$

iv.

$$\lim_{x \to -\infty} f(x) = 0$$