Math 721A1, Final Exam
Differential Topology I

(1) (15 points) Consider the smooth manifold $M := \{(v,w) \in \mathbb{R}^2 \,|\, w > 0\}$ with the covariant 2-tensor g on M given by
$$g = \frac{(dv)^2 + (dw)^2}{w^2}.$$
 (a) Prove that g is a Riemannian metric on M. The Riemannian manifold (M,g) is called the *Poincaré upper half plane.*
 (b) Consider a smooth curve $\gamma : \mathbb{R} \to M$ of the form $\gamma(t) := (0, f(t))$ where $f : \mathbb{R} \to \mathbb{R}_{>0}$ is a smooth function. Find f so that γ has unit speed with respect to the metric g.
 (c) Consider Riemannian manifold (D,h), called the *disk model*, where $D := \{(U,V) \in \mathbb{R}^2 \,|\, U^2 + V^2 < 1\}$ and
$$h = 4\frac{(dU)^2 + (dV)^2}{(1 - U^2 - V^2)^2}.$$
 Show that the map $\phi : M \to D$
$$\phi(v,w) = \left(\frac{v^2 + w^2 - 1}{v^2 + (w+1)^2}, \frac{-2v}{v^2 + (w+1)^2}\right)$$
 is an isometry.

(2) (15 points) Let $M = \mathbf{R}^3$ regarded as a smooth 3 dimensional manifold with Cartesian coordinates (x^1, x^2, x^3). Let $g = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ be the Euclean metric. Define the *standard volume form* ν in $\Omega^3(M)$ via
$$\nu := dx^1 \wedge dx^2 \wedge dx^3.$$
Given a vector field X, define α_X in $\Omega^1(M)$ by
$$\alpha_X := g(X, \cdot)$$
and define β_X in $\Omega^2(M)$ by
$$\beta_X := \nu(X, \cdot, \cdot).$$
 (a) For all functions f in $\Omega^0(M) = C^\infty(M)$, define the vector field ∇f on M via the equality of 1-forms
$$g(\nabla f, \cdot) := df.$$
 Show that ∇f agrees with the usual definition of the gradient vector field of f from vector calculus.
 (b) For all vector fields X, define the vector field $\nabla \times X$ via the equality of 2-forms
$$\nu(\nabla \times X, \cdot, \cdot) := d\alpha_X.$$
 Show that $\nabla \times X$ agrees with the usual definition of the curl of X from vector calculus.
 (c) For all vector fields X, define the smooth function $\nabla \cdot X$ via the equality of 3-forms
$$(\nabla \cdot X)\nu := d\beta_X.$$
 Show that $\nabla \cdot X$ agrees with the usual definition of the divergence of X from vector calculus.

(d) Show that the identities $\nabla \times \nabla f = 0$ and $\nabla \cdot \nabla \times X = 0$ are equivalent to the identities $d(df) = 0$ and $d(d\alpha_X) = 0$, respectively.

(3) (10 points) Problem 8-19 in Lee.
(4) (10 points) Problem 11-18 in Lee.
(5) (10 points) Problem 12-6 in Lee.
(6) (10 points) Problem 12-17 in Lee.
(7) (10 points) Problem 13-5 in Lee.