Math 124, Practice Exam Solutions for Exam #1, February 27, 2006

1. Calculate the following:

(a) Since $\sin^2(x) + \cos^2(x) = 1$, we obtain

$$I := \int \sin^{100}(x)\cos^3(x)\,dx = \int \sin^{100}(x)(1-\sin^2(x))\cos(x)\,dx = \int \sin^{100}(x)\cos(x)\,dx - \int \sin^{102}(x)\cos(x)\,dx.$$

Let $u = \sin x$ then $du = \frac{du}{dx}dx = \cos x dx$ then

$$I = \int u^{100}\,du - \int u^{102}\,du = \frac{u^{101}}{101} - \frac{u^{103}}{103} + C = \frac{\sin^{101}(x)}{101} - \frac{\sin^{103}(x)}{103} + C.$$

(b) Let $u = x^3 - 8x^2 + 5x + 3$ then $du = \frac{du}{dx}dx = (3x^2 - 16x + 5)\,dx$ then

$$\begin{aligned}\int \frac{3x^2 - 16x + 5}{\sqrt{x^3 - 8x^2 + 5x + 3}}\,dx &= \int (3x^2 - 16x + 5)(x^3 - 8x^2 + 5x + 3)^{-\frac{1}{2}}\,dx \\ &= \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} + C = 2(x^3 - 8x^2 + 5x + 3)^{\frac{1}{2}} + C\end{aligned}$$

(c) Let $u = x^2 - 3x$ then $\frac{du}{dx} = (2x - 3)$ or $\frac{1}{2}\frac{du}{dx} = (x - \frac{3}{2})\,dx$ and

$$\int (x - \frac{3}{2})\sin(x^2 - 3x)\,dx = \int \sin(u)\,\frac{1}{2}\,du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2 - 3x) + C$$

(d)

$$\int x\ln x^3\,dx = \int x(3\ln x)\,dx = 3\int x\ln x\,dx = 3\int u\,dv$$

where $u = \ln x$ and $v = \frac{x^2}{2}$. Using integration by parts, we obtain

$$\int x\ln x^3\,dx = 3\left(\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{d}{dx}(\ln x)\,dx\right) = \frac{3x^2}{2}\ln x - \frac{3x^2}{4} + C$$

(e) Let's use the substitution method. Let $\hat{u} = \sqrt{x}$ then $du = \frac{1}{2}x^{-\frac{1}{2}}dx$ or, in other words,

$$dx = 2x^{\frac{1}{2}}d\hat{u} = 2\hat{u}d\hat{u}$$

then

$$\int e^{\sqrt{x}}dx = \int e^{\hat{u}}2\hat{u}d\hat{u} = 2\int e^{\hat{u}}\hat{u}d\hat{u}.$$

Now we use integration by parts by setting $u = \hat{u}$ and $v = e^{\hat{u}}$ then

$$2\int e^{\hat{u}}\hat{u}d\hat{u} = 2(\hat{u}e^{\hat{u}} - \int e^{\hat{u}}d\hat{u}) = 2(\hat{u}e^{\hat{u}} - e^{\hat{u}}) + C = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

(f) Choose $u = \tan x$ then $du = \sec^2 x\,dx$. Also, when $x = 0$ then $u = \tan 0 = 0$ and when $x = \frac{\pi}{4}$ then $u = \tan\frac{\pi}{4} = 1$ so

$$\int_0^{\frac{\pi}{4}} \tan^5 x\sec^2 x\,dx = \int_0^1 u^5\,dx = \left.\frac{u^6}{6}\right|_0^1 = \frac{1}{6}$$

(g) Let $u = f(x)$ then $du = \frac{du}{dx}dx = f'(x)\,dx$. Therefore,

$$\int_2^4 f'(x)\sin(f(x))dx = \int_{f(2)}^{f(4)} \sin(u)\,du = \int_1^7 \sin(u)\,du = -\cos(7) + \cos(1).$$

(h) Notice that

$$\int (2x-8)e^{-x}\,dx \;=\; 2\int\, x\,e^{-x}\,dx - 8\int\, e^{-x}\,dx \;=\; 2\int x\,e^{-x}\,dx \,+\, 8e^{-x}$$

but by integration by parts,

$$\int x\,e^{-x}\,dx \;=\; -e^{-x} - xe^{-x} + C$$

therefore,

$$\int (2x-8)e^{-x}\,dx \;=\; 6e^{-x} - 2xe^{-x} \,+\, C$$

and

$$\int_1^3 (2x-8)e^{-x}\,dx \;=\; -\frac{4}{e}$$

(i)

$$\int_{-\infty}^{3} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_t^3 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}(\arctan(3)-\arctan(t)) = \arctan(3)-(-\frac{\pi}{2}) = \arctan(3)+\frac{\pi}{2}.$$

(j) This is an improper integral since the integrand is undefined when $x=\pm 2$ since $x^2-4=(x-2)(x+2)$. Therefore,

$$\int_1^5 \frac{x}{x^2-4}\,dx = \lim_{t\to 2^-}\int_1^t \frac{x}{x^2-4}\,dx + \lim_{t\to 2^+}\int_t^5 \frac{x}{x^2-4}\,dx$$

where the original integral converges if and only if both of the terms on the right hand side converge. Now,

$$\lim_{t\to 2^-}\int_1^t \frac{x}{x^2-4}\,dx = \lim_{t\to 2^-}(\frac{1}{2}\ln|t^2-4| - \frac{1}{2}\ln 3)$$

but this diverges. Therefore, the original integral diverges.

(k) The integral $\int_0^1 x^s \ln x\,dx$ is an improper integral. Thus,

$$\int_0^1 x^s \ln x\,dx = \lim_{t\to 0^+}\int_t^1 x^s \ln x\,dx.$$

The indefinite integral is evaluated by integration by parts where $u=\ln x$ and $dv = x^s dx$, or, $v=\frac{x^{s+1}}{s+1}$. (Notice we have used $s\neq -1$ here.) Thus,

$$\int x^s \ln x\,dx = (\ln x)(\frac{x^{s+1}}{s+1}) - \int \frac{x^{s+1}}{s+1}\frac{1}{x}\,dx = \frac{x^{s+1}}{s+1}\ln x - \frac{x^{s+1}}{(s+1)^2} + C$$

Therefore, plugging in the limits of integration, we have

$$\int_0^1 x^s \ln x\,dx = \lim_{t\to 0^+}\frac{-1+t^{1+s}-(1+s)\;t^{1+s}\;\ln(t)}{(1+s)^2}.$$

If $s<-1$ then the limit fails to exist and the original integral diverges. If $s>-1$ then the limit converges since the right hand side becomes

$$-\frac{1}{(s+1)^2} - \frac{1}{s+1}\lim_{t\to 0^+} t^{1+s}\ln t$$

but

$$\lim_{t\to 0^+} t^{1+s}\ln t = \lim_{t\to 0^+}\frac{\ln t}{t^{-1-s}} = \lim_{t\to 0^+}\frac{\frac{1}{t}}{(-1-s)t^{-2-s}} = \lim_{t\to 0^+}\frac{t^{s+1}}{-1-s} = 0.$$

where L'Hopital's rule has been used in the first equality and $s>-1$ has been used in the last equality.

(l) The average value by definition is

$$\overline{f} = \frac{1}{8-3}\int_3^8 x^2\,dx = \frac{97}{3}.$$

2. First, note that that two graphs intersect when $x^3 - x^5 = 0$ or, equivalently, when $x^3(1-x^2) = 0$ which occurs when $x = 0, \pm 1$. Therefore, since $x \geq 0$, R lies between $x = 0$ and $x = 1$.

(a) The area, A, of R is

$$A = \int_0^1 (x^3 - x^5)dx = \frac{1}{12}.$$

(b) The area of an equilateral triangle with a side of length s is

$$\frac{1}{2}\left(\frac{\sqrt{3}}{2}s\right)(s) = \frac{s^2\sqrt{3}}{4}.$$

The length of an edge of the triangle located at the position x (where $0 \leq x \leq 1$) is $x^3 - x^5$. Therefore, the area of such a triangle is thus

$$A(x) = \frac{\sqrt{3}}{4}(x^3 - x^5)^2.$$

The volume of S is

$$V = \int_0^1 A(x)dx = \int_0^1 \frac{\sqrt{3}}{4}(x^3 - x^5)^2 dx = \frac{\sqrt{3}}{4}\int_0^1 (x^6 - 2x^8 + x^{10})dx.$$

Performing the latter integral, one obtains

$$V = \frac{\sqrt{3}}{4}(\frac{1}{7} - 2\left(\frac{1}{9}\right) + \frac{1}{11}) = \frac{2\sqrt{3}}{693}.$$

3. Since $y = 9 - x^2 = (3-x)(3+x)$, $y = 0$ when $x = \pm 3$. The area of R is thus

$$A = \int_{-3}^3 (9 - x^2)\,dx = 36.$$

Thus

$$\overline{x} = \frac{1}{A}\int_{-3}^3 x(9-x^2)\,dx = 0$$

where the last equality is because $x(9-x^2)$ is an odd function. However,

$$\overline{y} = \frac{1}{2A}\int_{-3}^3 (9-x^2)^2\,dx = \frac{1}{72}\int_{-3}^3 (81 - 18x^2 + x^4)\,dx = \frac{18}{5}.$$

Therefore, the centroid of R is $(\overline{x}, \overline{y}) = (0, \frac{18}{5})$.

4. The arc length L is given by

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

but

$$\frac{dy}{dx} = 3x^{\frac{1}{2}}.$$

Plugging this into the arc length formula, we need to calculate the integral

$$L = \int_1^4 \sqrt{1+9x}\,dx$$

but by using substitution, we can see that

$$\int \sqrt{1+9x}dx = \frac{2}{27}(1+9x)^{\frac{3}{2}}$$

Plugging in, we obtain

$$L = \frac{-20\sqrt{10}}{27} + \frac{74\sqrt{37}}{27}.$$

5. The graphs intersect when $0 = (x-2)^2 - x = x^2 - 5x + 4 = (x-4)(x-1)$; in other words, at $x = 1, 4$.

   The volume V of the region is

$$V = \int_1^4 (\pi x^2 - \pi((x-2)^2)^2)\, dx$$

   but

$$\int (\pi x^2 - \pi((x-2)^2)^2)\, dx = -16\,\pi\,x + 16\,\pi\,x^2 - \frac{23\,\pi\,x^3}{3} + 2\,\pi\,x^4 - \frac{\pi\,x^5}{5}.$$

   Plugging in the limits, we obtain

$$V = \frac{72}{5}\pi.$$

6. (a) Just calculate the following:

$$\frac{d}{dx}\left(Ce^{-2x} + \frac{1}{3}e^x\right) + 2\left(Ce^{-2x} + \frac{1}{3}e^x\right) = -2Ce^{-2x} + \frac{1}{3}e^x + 2Ce^{-2x} + \frac{2}{3}e^x = e^x$$

   (b) Just plug into the general solution

$$8 = y(0) = C + \frac{1}{3}$$

   then

$$C = 8 - \frac{1}{3} = \frac{23}{3}.$$

7. (a) The equation can be rewritten as

$$\frac{dy}{dx} = -3\frac{\cos x}{y^2}$$

   which is separable. Therefore, one obtains

$$\int y^2\, dy = \int -3\cos x\, dx$$

   or

$$\frac{y^3}{3} = -3\sin x + K$$

   where K is a constant. Solving for y, one obtains

$$y = (C - 9\sin x)^{\frac{1}{3}}.$$

   where C is a constant.

   (b)

$$2 = y(\pi) = C^{\frac{1}{3}}$$

   Hence, $C = 8$. The particular solution is

$$y = (8 - 9\sin x)^{\frac{1}{3}}.$$

8. Taking derivatives of $y = \frac{k}{x}$, we obtain

$$\frac{dy}{dx} = -\frac{k}{x^2} = -\frac{xy}{x^2} = -\frac{y}{x}$$

where we have used that $xy = k$ in the second equality. Therefore, orthogonal trajectories satisfy the equation

$$\frac{dy}{dx} = -\frac{1}{-\frac{y}{x}} = \frac{x}{y}.$$

Separating variables, we obtain the equality

$$\int y dy = \int x dx$$

which becomes

$$\frac{y^2}{2} = \frac{x^2}{2} + \tilde{C}$$

for an arbitrary constant $\tilde{C}$ which we can rewrite as

$$y^2 = x^2 + C$$

where C is an arbitrary constant.