Math 124, Solutions to Practice Questions for Exam #2, April 24, 2006

1. Which of the following series converges? Explain your answer.

(a)

$$\frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \cdots$$

This is an alternating series which can be written as

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}.$$

Since $\frac{1}{\ln(n+1)} > \frac{1}{\ln(n+2)}$ for all n and

$$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0,$$

the series converges by the Alternating Series Test.

(b)

$$\sum_{n=1}^{\infty} \frac{n}{(\sin n)^2}$$

Since $(\sin n)^2 \leq 1$ for all n, $\frac{n}{(\sin n)^2} \geq n$ for all n. Taking limits of both sides

$$\lim_{n\to\infty} \frac{n}{(\sin n)^2} \geq \lim_{n\to\infty} n = \infty$$

which is nonzero. Therefore, the series diverges by the Divergence Test.

(c)

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Consider the function $f(x) = \frac{1}{x(\ln x)^2}$. It is certainly positive when $x \geq 2$. It is decreasing since its numerator is constant and its denominator is increasing. Thus, the function $f(x)$ is positive and decreasing on the interval $x \geq 2$. Furthermore,

$$\lim_{x\to\infty} \frac{1}{x(\ln x)^2} = 0$$

hence, we can use the Integral Test which states that the series converges if and only if $\int_2^\infty f(x)dx$ converges. The latter is

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \left(-\frac{1}{\ln t} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Therefore, the series converges.

(d)

$$\sum_{n=1}^{\infty} \frac{n^2}{8n^7 + 6n^2 + 5}$$

Use the Limit Comparision Test. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^5}.$$

Both series have positive terms and, in addition,

$$\lim_{n\to\infty} \frac{\frac{n^2}{8n^7+6n^2+5}}{\frac{1}{n^5}} = \frac{1}{8}.$$

By the Limit Comparision Test, both series either converge or both series diverge. However, $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges as it is a p-series with $p = 5 > 1$.

(e)

$$\sum_{n=1}^{\infty} 2^{2n+1} 5^{-n}$$

Notice that

$$\sum_{n=1}^{\infty} 2^{2n+1} 5^{-n} = \sum_{n=1}^{\infty} (2)(\frac{4}{5})^n = \sum_{n=1}^{\infty} (\frac{8}{5})(\frac{4}{5})^{n-1}$$

which is a geometric series which converges since $|\frac{4}{5}| < 1$. In fact, we even know that the series is equal to

$$\frac{\frac{8}{5}}{1-\frac{4}{5}} = 8.$$

(f)

$$\sum_{n=1}^{\infty} a_n$$

where $a_1 = 7$ and $a_{n+1} = \frac{2n+3}{5n+9} a_n$ for all $n \geq 1$.
Let's use the Ratio Test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2n+3}{5n+9}\right| = \frac{2}{5}$$

but $\frac{2}{5} < 1$, therefore, the series converges.

2. Consider the following series

$$s = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+n^2}$$

How many terms in the series must one sum up in order to obtain s correct to within 0.000001 accuracy?

Since s is an alternating series satisfying $\frac{1}{1+(n+1)^2} < \frac{1}{1+n^2}$ for all n and

$$\lim_{n\to\infty} \frac{1}{1+n^2} = 0,$$

the Alternating Series Estimation Theorem tells us that the remainder $R_n = s - s_n$ where $s_n = \sum_{k=0}^{n} \frac{(-1)^k}{1+k^2}$ satisfies

$$|R_n| \leq \frac{1}{1+(n+1)^2}$$

for all n. We want

$$\frac{1}{1+(n+1)^2} < 0.000001 = 10^{-6}$$

which is equivalent to $n > \sqrt{99999} - 1 \cong 998.999$. Therefore, s_{999} is equal to s to within an accuracy 0.000001. So we need sum up the first 1000 terms of s to obtain the desired accuracy.

3. Consider the following series

$$s = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

How many terms in the series must one sum up in order to obtain s correct to within an accuracy of 0.00001?

Recall that the series s converges by the Integral Test. By the Remainder Estimate for the Integral Test, the remainder $R_n = s - s_n$ where $s_n = \sum_{k=1}^{n} \frac{1}{k^3}$ satisfies

$$R_n \leq \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2}.$$

We want

$$\frac{1}{2n^2} < 0.00001$$

which is equivalent to $n > \sqrt{5000} \cong 223.607$. Therefore, s_{224} (which is the sum of the first 224 terms) is equal to s up to an accuracy of 0.00001.

4. Suppose that we know that a power series $\sum_{n=0}^{\infty} c_n(x-1)^n$ converges when $x = 4$ and diverges at $x = -6$.

We know that for any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there exists a number R, called the radius of convergence, such that if $|x-a| < R$ then the series converges, if $|x-a| > R$ then the series diverges, and if $|x-a| = R$ then the series could either converge or diverge.

In our situation $a = 1$, clearly. Since the series converges for $x = 4$, it must be true that $|4-1| = 3 \leq R$. On the other hand, the series diverges for $x = -6$ so $|-6-1| = 7 \geq R$. So the only thing we know is that

$$3 \leq R \leq 7.$$

(a) What can one say about the convergence of the series at $x = -1$?
Since $|-1-1| = 2 < R$, the series converges at $x = -1$.

(b) What can one say about the convergence of the series at $x = -7$?
Since $|-7-1| = 8 > R$, the series diverges at $x = -7$.

(c) What can one say about the convergence of the series at $x = -2$?
Since all we know is that $|-2-1| = 3 \leq R$, the series could either diverge or converge since R could be equal to 3. So we can't say anything without any more information.

5. Consider the function $f(x) = \frac{3x^4}{5x-7}$.

(a) Write $f(x)$ as a power series.

$$\frac{3x^4}{5x-7} = -\frac{3x^4}{7}\frac{1}{1-\frac{5x}{7}} = -\frac{3x^4}{7}\sum_{n=0}^{\infty}(\frac{5x}{7})^n$$

using $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ and plugging in $u = \frac{5x}{7}$. Multiplying though, one obtains

$$f(x) = \sum_{n=0}^{\infty} -\frac{(3)5^n}{7^{n+1}} x^{n+4}.$$

(b) Find its radius of convergence.
Its radius of convergence, R, is equal to the radius of convergence of $\sum_{n=0}^{\infty}(\frac{5x}{7})^n$. But the latter converges if and only if $|\frac{5x}{7}| < 1$ or, equivalently, when $|x| < \frac{7}{5}$. Therefore, $R = \frac{7}{5}$.

(c) Find its interval of convergence.
The series converges on the interval $(-\frac{7}{5}, \frac{7}{5})$.

6. Consider the function $f(x) = \tan^{-1}(x^3)$.

(a) Write $f(x)$ as a power series.
We have that

$$\tan^{-1} u = \int \frac{1}{1+u^2} du = \int \sum_{n=0}^{\infty} (-u^2)^n du = C + \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{2n+1}$$

where C is an integration constant. However, since $\tan^{-1}(0) = 0$, $C = 0$. Therefore,

$$\tan^{-1} u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{2n+1}.$$

Now we just set $u = x^3$ to obtain

$$\tan^{-1} x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}.$$

(b) Find its radius of convergence.
The series converges if $|u| = |x^3| < 1$ which is equivalent to $|x| < 1$. Therefore, the radius of convergence is 1.

7. Consider the series $\sum_{n=1}^{\infty} \frac{x^n}{2^n n}$.

(a) Find its radius of convergence.
Since

$$\lim_{n\to\infty} \left| \frac{\frac{x^{(n+1)}}{2^{(n+1)}(n+1)}}{\frac{x^n}{2^n n}} \right| = \lim_{n\to\infty} \frac{|x|}{2} \frac{n}{n+1} = \frac{|x|}{2},$$

the series converges if $|x| < 2$ and diverges if $|x| > 2$ by the Ratio Test. Therefore, the radius of convergence is 2.

(b) Find its interval of convergence.
We need only check $x = \pm 2$. If $x = 2$ then the series is

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is the Harmonic series and, hence, diverges. If $x = -2$ then the series is

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which converges by the Alternating Series Test. Therefore, the interval of convergence is $[-2, 2)$.

8. Find the Taylor series centered at 1 of the function $f(x) = x^{\frac{2}{3}}$.

Notice that $f^{(0)}(x) = x^{\frac{2}{3}}$ and for all $n \geq 1$,

$$f^{(n)}(x) = (\frac{2}{3})(\frac{2}{3} - 1) \cdots (\frac{2}{3} - n + 1) x^{\frac{2}{3} - n}.$$

The we have

$$\sum_{n=0}^{\infty} f^{(n)}(1) \frac{(x-1)^n}{n!} = \sum_{n=0}^{\infty} c_n \frac{(x-1)^n}{n!}$$

where $c_n = (\frac{2}{3})(\frac{2}{3} - 1) \cdots (\frac{2}{3} - n + 1)$ if $n \geq 1$ and $c_0 = 1$.

9. (a) Find the MacLauren series of the function $f(x) = \ln(3+x)$.
Notice that $f^{(0)}(x) = \ln(3+x)$ and for all $n \geq 1$,

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(3+x)^{-n}.$$

Plugging in 0, we obtain

$$\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} = \ln 3 + \sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!3^{-n}\frac{x^n}{n!} = \ln 3 + \sum_{n=1}^{\infty}(-1)^{n-1}3^{-n}\frac{x^n}{n}.$$

(b) Find its radius of convergence.
Since

$$\lim_{n\to\infty}\left|\frac{(-1)^n 3^{-(n+1)}\frac{x^{n+1}}{n+1}}{(-1)^{n-1}3^{-n}\frac{x^n}{n}}\right| = \lim_{n\to\infty}\frac{|x|}{3}\frac{n}{n+1} = \frac{|x|}{3},$$

the Ratio Test implies that the radius of convergence is 3.

10. (a) Find a power series expression for the following integral:

$$\int e^{-x^4}\,dx$$

We use the formula

$$e^u = \sum_{n=0}^{\infty}\frac{u^n}{n!}$$

then setting $u = -x^4$,

$$\int e^{-x^4}\,dx = \int (\sum_{n=0}^{\infty}\frac{(-x^4)^n}{n!})\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\int x^{4n}\,dx) = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{4n+1}}{4n+1}$$

where C is an integration constant.

(b) Find a series representation for the following:

$$\int_0^2 e^{-x^4}\,dx$$

Just take the answer in the previous answer, plug in $x = 2$ and $x = 0$ and take the difference to obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{2^{4n+1}}{4n+1}$$

11. Calculate

$$\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!\,2^{2n}}$$

Recall the power series expansion for $\sin x$ which is

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Plugging in $x = \frac{\pi}{2}$, we obtain

$$\sin\frac{\pi}{2} = \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!\,2^{2n+1}}.$$

Multiplying through by 2, we obtain

$$2\sin\frac{\pi}{2} = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}2}{(2n+1)!\,2^{2n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!\,2^{2n}}$$

which is our desired series. The answer is therefore $2\sin\frac{\pi}{2} = 2$.